Symmetry and Exact Solutions of the Maxwell and $SU(2)$ Yang-Mills Equations

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Abstract

We give the overview of solution techniques for the general conformally-invariant linear and nonlinear wave equations centered around the idea of dimensional reductions by their symmetry groups. The efficiency of these techniques is demonstrated on the examples of the $SU(2)$ Yang-Mills and the vacuum Maxwell equations. For the Yang-Mills equations we have derived the most general form of the conformally-invariant solution and construct a number of their new analytical non-Abelian solutions in explicit form. We have completely solved the problem of symmetry reduction of the Maxwell equations by subgroups of the conformal group. This yields twelve multi-parameter families of their exact solutions, a majority of which are new and might be of considerable interest for applications.

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1 Introduction.

The famous paper [1] written by Yang and Mills is a milestone of the modern quantum physics, where the role played by the equations introduced in the paper (called now the $SU(2)$ Yang-Mills equations) can be compared only to that of the Klein-Gordon-Fock, Schrödinger, Maxwell and Dirac equations. However, the real importance of the Yang-Mills equations has been understood only in the late sixties, when the concept of the gauge fields as of those responsible for all the four fundamental physical interactions (gravitational, electro-magnetic, weak and strong interactions) has become widely spread.

The simplest example of the gauge theory in the $(1+3)$-dimensional space is the system of the Maxwell equations for the four-component vector-potential of the electro-magnetic field, whose gauge group is the one-parameter group $U(1)$. The simplest example of the non-Abelian gauge group is the group $SU(2)$. This very group is realized as the symmetry group admitted by the Yang-Mills equations describing the triplet of the gauge fields (called in the sequel the Yang-Mills field) $\mathbf{A}_\mu(x) = (A^a_\mu(x), a = 1, 2, 3)$, where $\mu = 1, 2, 3, 4,\ x = (x_1, x_2, x_3, x_4)$ for the case of the four-dimensional Euclid space and $\mu = 0, 1, 2, 3,\ x = (x_0 = t, x_1, x_2, x_3) = (t, \mathbf{x})$ for the case of the Minkowski space. The matrix vector field $A_\mu = A_\mu(x)$ is defined as follows:

$$A_\mu = e\frac{\sigma_a}{2i}A^a_\mu.$$

Here σ_a, $(a = 1, 2, 3)$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and e is the real constant called the gauge coupling constant.

Using the matrix gauge potentials one constructs the matrix-valued field

$$F_{\mu\nu} \equiv \partial^\mu A_\nu - \partial^\nu A_\mu + [A_\mu,\ A_\nu], \quad \mu, \nu = 0, 1, 2, 3.$$

Writing the above expressions component-wise yields

$$F_{\mu\nu} \equiv e\frac{\sigma_a}{2i}F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial^\mu A^a_\nu - \partial^\nu A^a_\mu + e f^a_{bc} A^b_\mu A^c_\nu,$$

where $\mu, \nu = 0, 1, 2, 3$, $a = 1, 2, 3$ and the symbols f^a_{bc}, $(a, b, c = 1, 2, 3)$ stand for the structure constants determining the Lie algebra of the gauge group (note that for the case of the group $SU(2)$, $f^a_{bc} = \varepsilon_{abc}$, ε_{abc} being the anti-symmetric tensor with $\varepsilon_{123} = 1$, $a, b, c = 1, 2, 3$).

Hereafter we use the following designations:

$$\partial_\mu = \partial_{x_\mu} = \frac{\partial}{\partial x_\mu}.$$

Furthermore, lowering and rising the indices μ, ν is performed with the help of the metric tensor of the space of the variables x_μ and the summation over the repeated indices is carried out.

The $SU(2)$ Yang-Mills equations are obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

and are of the form

$$\partial_\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = [D_\mu, F_{\mu\nu}] = 0, \tag{1.1}$$

where $D_\mu = \partial_\mu + A^\mu$ is the covariant derivative, $\mu, \nu = 0, 1, 2, 3$.

One of the most popular and exciting parts of the general theory of the Yang-Mills equations is that devoted to constructing their exact analytical solutions. There is a vast literature devoted solely to constructing and analyzing exact solutions of (1.1) (see, the review by Actor [2] and the monograph [3] for the extensive list of references). The majority of the results is obtained for the case of the Yang-Mills equations in the Euclidean space. The principal reason for this is the fact that equations (1.1) in the Euclidean space have the monopole and instanton solutions [3, 4], that admit numerous physical interpretations and have highly non-trivial geometrical and algebraic properties. Note that these and some other classes of exact solutions of system (1.1) can be also obtained by solving the so-called self-dual Yang-Mills equations

$$F_{\mu\nu} = *F_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3. \tag{1.2}$$

Evidently, each solution of (1.2) satisfies (1.1), while the reverse assertion does not hold.

Provided we consider the Euclidean case, $*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}$, $(\mu, \nu, \lambda, \rho = 1, 2, 3, 4)$, where $\varepsilon_{\mu\nu\lambda\rho}$ is the completely anti-symmetric tensor, and equations (1.2) form the system of four real first-order partial differential equations.

It was the self-duality property of the instanton solutions of (1.1) in the Euclidean space, that had enabled using the ansatz, suggested by t'Hooft [5], Corrigan and Fairlie [6], Wilczek [7] and Witten [8], in order to construct these solutions. Furthermore, the well-known monopole solution by Prasad and Sommerfield [9], as well as, the solutions obtainable via the Atyah-Hitchin-Drinfeld-Manin method [10] exploit explicitly the self-duality condition.

One more important property of the self-dual Yang-Mills equations is that they are equivalent to the compatibility conditions of some over-determined system of linear partial differential equations [11, 12]. In other words, the self-dual Yang-Mills equations admit the Lax representation and, in this sense, are integrable. By this very reason it is possible to reduce equations (1.2) to the well-studied solitonic equations, such as the Euler-Arnold, Burgers and

Devy-Stuardson equations (Chakravarty et al, [13, 14]) and Liouville and sine-Gordon equations (Tafel, [15]) by the use of the symmetry reduction method.

For the case, when the Yang-Mills field are defined in the Minkowski space, we have, $*F_{\mu\nu} = \frac{i}{2}\, \varepsilon_{\mu\nu\lambda\rho}\, F^{\lambda\rho},\ (\mu, \nu, \lambda, \rho = 0, 1, 2, 3)$. Consequently, equations (1.2) form the system of complex first-order differential equations. In view of this fact, exploitation of the above mentioned methods and results for study of the $SU(2)$ Yang-Mills equations (1.1) in the Minkowski space yields complex-valued solutions. That is why, the above mentioned methods for solving equations (1.1) fail to be efficient for the case of the Minkowski space. Consequently, there is a need for developing the new methods, that do not rely on the self-duality condition. This problem has been addressed by one of the creators of the inverse scattering technique V.E.Zaharov, who wrote in the foreword to the Russian translation of the monograph by Calogero and Degasperis [16], that a number of important problems of the nonlinear mathematical physics (including the Yang-Mills equations in the Minkowski space) are still waiting for new efficient solution techniques to appear.

On the other hand, it is known [17] (see, also, [18]) that equations (1.1) have rich symmetry. Namely, their maximal (in the Lie sense) symmetry group is the group $G \otimes SU(2)$, where G is

- the conformal group $C(1, 3)$, if the Yang-Mills equations are defined in the Minkowski space;

- the conformal group $C(4)$, if the Yang-Mills equations are defined in the Euclidean space;

- the conformal group $C(2, 2)$, if the Yang-Mills equations are defined in the pseudo-Euclidean space having the metric tensor with the signature $(-,\ -,\ +,\ +)$.

Note that the maximal symmetry groups admitted by the self-dual Yang-Mills equations (1.2) coincide with the symmetry groups of the corresponding equations (1.1).

The rich symmetry of equations (1.1), (1.2) enables efficient exploitation of the symmetry reduction routine for the sake of dimensional reduction of the Yang-Mills equations either to ordinary differential equations integrable by quadratures or to integrable solitonic equations in two or three independent variables [19]–[21]. In particular, some subgroups of the generalized Poincaré group $P(2, 2)$, which is the subgroup of the conformal group $C(2, 2)$, were used in order to reduce the self-dual Yang-Mills equations, defined in the pseudo-Euclidean space having the metric tensor with the signature $(-,$ $-,\ +,\ +)$, to a number of known integrable systems, like, the Ernst, cubic

Schrödinger and Euler-Calogero-Moser equations (see, [22] and the references therein). Legaré et al have carried out systematic investigation of the problem of symmetry reduction of system (1.2) in the Euclidean space by subgroups of the Euclid group $E(4) \in C(4)$ [23, 24]. What is more, some of the known analytical solutions of equations (1.1) in the Euclidean space (namely, the non self-dual meron solution, obtained by Alfaro, Fubini and Furlan [25], and the instanton solution, constructed by Belavin, Polyakov, Schwartz and Tyupkin [26]) can also be obtained within the framework of the symmetry reduction approach (see, e.g., [21]).

To the best of our knowledge, the first paper devoted to symmetry reduction of the $SU(2)$ Yang-Mills equations in the Minkowski space has been published by Fushchych and Shtelen [27] (see, also, [21]). They use two conformally-invariant ansatzes in order to perform reduction of equations (1.1) to systems of ordinary differential equations. Integrating the latter yields several exact solutions of the Yang-Mills equations (1.1).

Let us note that the full solution of the problem of symmetry reduction of fundamental equations of relativistic physics, whose symmetry groups are subgroups of the conformal group $C(1,3)$, has been obtained for the scalar wave equation only (see, for further details, [21], [28]–[30]). This fact is explained by the extreme cumbersomity of the calculations needed to perform a systematic symmetry reduction of systems of partial differential equations by all inequivalent subgroups of the conformal group $C(1,3)$. The complete solution of the problem symmetry reduction to systems of ordinary differential equations has been obtained for the conformally-invariant nonlinear spinor equations [31]–[33], that generalize the Dirac equation for an electron. In our recent publications we have carried out symmetry reduction of the Yang-Mills equations (1.1), (1.2) by subgroups of the Poincaré group and have constructed a number of their exact solutions [34]–[39].

The principal aim of the present paper is two-fold. Firstly, we will review the already known ideas, methods and results, centered around the solution techniques, that are based on the symmetry reduction method for the Yang-Mills equations (1.1), (1.2) in the Minkowski space. Secondly, we will expose the general reduction routine, developed by us recently, that enables the unified treatment of both the classical and non-classical symmetry reduction approaches for an arbitrary relativistically-invariant system of partial differential equations. As a by-product, this approach yields exhaustive solution of the problem of symmetry reduction of the vacuum Maxwell equations

$$\begin{aligned} \operatorname{rot}\mathbf{E} &= -\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \\ \operatorname{rot}\mathbf{H} &= \frac{\partial \mathbf{E}}{\partial t}, \qquad \operatorname{div}\mathbf{E} = 0. \end{aligned} \tag{1.3}$$

The history of the study of symmetry properties of equations (1.3) goes back to the beginning of the century. Invariance properties of the Maxwell

equations have been studied by Lorentz [40] and Poincaré [41, 42]. They have proved that equations (1.3) are invariant with respect to the transformation group named by the Poincaré's suggestion the Lorentz group. Furthermore, Larmor [43] and Rainich [44] have found that equations (1.3) are invariant with respect the one-parameter transformation group

$$\mathbf{E} \to \mathbf{E}\cos\theta + \mathbf{H}\sin\theta, \quad \mathbf{H} \to \mathbf{H}\cos\theta - \mathbf{E}\sin\theta \tag{1.4}$$

called now the Heviside-Larmor-Rainich group. Later on, Bateman [45] and Cunningham [46] showed that the Maxwell equations are invariant with respect to the conformal group.

Much later, Ibragimov [47] have proved that the group $C(1,3)\otimes H$, where $C(1,3)$ is the group of conformal transformations of the Minkowski space and H is the Heviside-Larmor-Rainich group (1.4), is the maximal in Lie's sense invariance group of equations (1.3). Note that this result coincides with that obtained earlier without explicit use of the infinitesimal Lie algorithm [19, 20]. A further progress in study of symmetries of the Maxwell equations has become possible, when Fushchych and Nikitin suggested the non-Lie approach to investigating symmetry properties of linear systems of partial differential equations (see, for more details, [48]).

The present review is based mainly on our publications [33]–[35], [36, 37, 38, 39], [49]–[53] and has the following structure. In the second section we give the detailed description of the general reduction routine for an arbitrary relativistically-invariant systems of partial differential equations. The results of this section are used in the third one in order to solve the problem of symmetry reduction of the Yang-Mills equations (1.1) by subgroups of the Pojncaré group $P(1,3)$ and to construct their exact (non-Abelian) solutions. In the next section we review the techniques for non-classical reductions of the $SU(2)$ Yang-Mills equations, that are based on their conditional symmetry. These techniques enable obtaining the principally new classes of exact solutions of (1.1), that are not derivable within the framework of the standard symmetry reduction technique. In the fifth section we give an overview of the known invariant solutions of the Maxwell equations and construct multi-parameter families of new ones.

2 Conformally-invariant ansatzes for an arbitrary vector field

In this section we describe the general approach to constructing conformally-invariant ansatzes applicable to any (linear or non-linear) system of partial differential equations, on whose solution set a linear covariant representation of the conformal group $C(1,3)$ is realized. Since the majority of the equations of the relativistic physics, including the Klein-Gordon-Fock, Maxwell,

massless Dirac and Yang-Mills equations respect this requirement, they can be handled within the framework of this approach.

Note that all our subsequent considerations are local and the functions involved are supposed to be as many times continuously differentiable, as it is necessary for performing the corresponding mathematical operations.

2.1 On the linear form of invariant ansatzes

Consider the system of partial differential equations (we denote it as S)

$$S : \underset{A}{F}(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, \ldots, \underset{r}{\mathbf{u}}) = 0, \quad A = 1, \ldots, m, \tag{2.1}$$

defined on the open subset $M \subset X \times U \simeq R^q \times R^p$ of the space of p independent and q dependent variables. In (2.1) we use the notations, $\mathbf{x} = (x_1, \ldots, x_p) \in X$, $\mathbf{u} = (u^1, \ldots, u^q) \in U$, $\underset{l}{\mathbf{u}} = \Big\{ \dfrac{\partial^l u^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_p^{\alpha_p}},$ $0 \le \alpha_i \le l,\ \sum_{i=1}^p \alpha_i = l,\ k)1, \ldots, q \Big\}$, $l = 1, 2, \ldots, r$ and F_A are sufficiently smooth functions of the given arguments.

Let G be a local transformation group, that acts on M and is the symmetry group of system (2.1). Next, let the basis operators of the Lie algebra g of the group G be of the form

$$X_a = \xi_a^i(\mathbf{x}, \mathbf{u})\partial_{x_i} + \eta_j^a(\mathbf{x}, \mathbf{u})\partial_{u^j},\ a = 1, \ldots, n, \tag{2.2}$$

where ξ_a^i, η_j^a are arbitrary smooth functions on M, $\partial_{u^j} = \frac{\partial}{\partial u^j}$, $i = 1, \ldots, p, j = 1, \ldots, q$. By definition, operators (2.2) satisfy the commutation relations

$$[X_a, X_b] \equiv X_a X_b - X_b X_a = C_{ab}^c X_c,\ a, b, c = 1, \ldots, n,$$

where C_{ab}^c are the structure constants, that determine uniquely the type of the Lie algebra g.

We say that a solution $\mathbf{u} = \mathbf{f}(\mathbf{x})$, ($\mathbf{f} = (f^1, \ldots, f^q)$) of system (2.1), is G-invariant, if the manifold $\mathbf{u} - \mathbf{f}(\mathbf{x}) = 0$ is invariant with respect to the action of the group G. This means that for an arbitrary $g \in G$ the functions $\mathbf{f}$ and $g(\mathbf{f})$ coincide in the intersection of the domains, where they are defined. More precisely, we can define G-invariant solution of system (2.1) as the solution $\mathbf{u} = \mathbf{f}(\mathbf{x})$, whose graph $\Gamma_{\mathbf{f}} = \{(\mathbf{x}, \mathbf{f}(\mathbf{x}))\} \subset M$ is locally G-invariant subset of the set M.

If G is the symmetry group of system (2.1), then, under some additional assumption of regularity of the action of the group G, we can find all its G-invariant solutions by solving the reduced system of differential equations S/G. Note that by construction the system S/G has fewer number of independent variables, i.e., the dimension of the initial system is reduced (by this very reason, the above procedure is called the symmetry reduction method).

In the sequel, we will restrict our considerations to the case of the projective action of the group G in M. This means that all the transformations g from G are of the form

$$(\overline{\mathbf{x}}, \overline{\mathbf{u}}) = g((\mathbf{x}, \mathbf{u})) = (\Psi_g(\mathbf{x}), \Phi_g(\mathbf{x}, \mathbf{u})).$$

In other words, the transformation law for the independent variables $\mathbf{x}$ does not involve the dependent variables (for the Lie algebra g of the group G this implies that in formulae (1.2) $\xi_a^i = \xi_a^i(\mathbf{x})$). This defines the projective action of the group G $\overline{\mathbf{x}} = g(\mathbf{x}) = \Psi_g(\mathbf{x})$ in an arbitrary subset Ω of the set X.

In what follows, we will suppose that the action of the group G in M and its projective action in Ω are regular and the orbits of these actions have the same dimension s. This dimension is called the rank of the group G (or, alternatively, the rank of the Lie algebra g). Note that the condition $\operatorname{rank} G = s$ is equivalent to the requirement that the relation

$$\operatorname{rank} \parallel \xi_a^i(\mathbf{x}_0) \parallel = \operatorname{rank} \parallel \xi_a^i(\mathbf{x}_0), \eta_j^a(\mathbf{x}_0, \mathbf{u}_0) \parallel = s \tag{2.3}$$

holds in an arbitrary point $(\mathbf{x}_0, \mathbf{u}_0) \in M$ [19]. And what is more, we will suppose that $s < p$ (the case $s = p$ is trivial, and furthermore, G-invariant functions do not exist under $s > p$).

If the above assumptions hold, then there are $p - s$ fuctionally independent invariants $y^1 = \omega^1(\mathbf{x}), y^2 = \omega^2(\mathbf{x}), \ldots, y^{p-s} = \omega^{p-s}(\mathbf{x})$ (the first set of invariants) of the group G acting projectively in Ω, and what is more, each of them is the invariant of the group G acting in M. Furthermore, there are q functionally independent invariants $v^1 = g^1(\mathbf{x}, \mathbf{u})$, $v^2 = g^2(\mathbf{x}, \mathbf{u}), \ldots, v^q = g^q(\mathbf{x}, \mathbf{u})$ of the group G acting in M (the second set of invariants) [19, 20]. Using the short-hand notation we represent the full set of invariants of the group G in the following way:

$$\mathbf{y} = \mathbf{w}(\mathbf{x}), \quad \mathbf{v} = \mathbf{g}(\mathbf{x}, \mathbf{u}). \tag{2.4}$$

Owing to the validity of the relation

$$\operatorname{rank} \left|\left| \frac{\partial g^j}{\partial u^i} \right|\right| = q, \quad i, j = 1, \ldots, q$$

we can solve locally the second system of equations from (2.4) with respect to $\mathbf{u}$

$$\mathbf{u} = \tilde{\mathbf{r}}(\mathbf{x}, \mathbf{v}). \tag{2.5}$$

Using the relation

$$\operatorname{rank} \left|\left| \frac{\partial \omega^j}{\partial x_i} \right|\right| = p - s, \quad j = 1, \ldots, p - s, \quad i = 1, \ldots, p,$$

we choose $p-s$ independent variables $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_{p-s})$ so that

$$\operatorname{rank} \left|\left| \frac{\partial \omega^j}{\partial \tilde{x}_i} \right|\right| = p - s, \quad i, j = 1, \dots, p-s.$$

We call these variables principal. The remaining s independent variables $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_s)$ are called parametrical (they enter all the subsequent formulae as parameters).

Now we can solve the first system from (2.4) with respect to the principal variables

$$\tilde{\mathbf{x}} = \mathbf{z}(\hat{\mathbf{x}}, \mathbf{y}). \tag{2.6}$$

Inserting (2.6) into (2.5) we get the equality

$$\mathbf{u} = \tilde{\mathbf{r}}(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{v})$$

or

$$\mathbf{u} = \mathbf{r}(\hat{\mathbf{x}}, \mathbf{y}, \mathbf{v}). \tag{2.7}$$

Note that in (2.5)–(2.7), $\tilde{\mathbf{r}} = (\tilde{r}^1, \dots, \tilde{r}^q)$, $\mathbf{r} = (r^1, \dots, r^q)$, $\mathbf{z} = (z^1, \dots, z^{p-s})$. The so constructed G-invariant function (2.7) is called the *ansatz*. Inserting ansatz (2.7) into system (2.1) yields the system of partial differential equations for the functions $\mathbf{v}$ of the variables $\mathbf{y}$, that do not involve explicitly the parametrical variables [19]. These equations form the reduced (or factor) system S/G having the fewer number of independent variables $y^1, \dots, y^{p-s}$, as compared with the initial system (2.1). Now, if we are given a solution $\mathbf{v} = \mathbf{h}(\mathbf{y})$ of the reduced system, then inserting it into (2.7) yields a G-invariant solution of system (2.1).

Summing up, we formulate the algorithm of symmetry reduction and construction of invariant solutions of systems of partial differential equations, that admit non-trivial Lie symmetry.

(I) Using the infinitesimal Lie method we compute the maximal symmetry group G admitted by the equation under study.

(II) We fix the symmetry degree s of the invariant solutions to be constructed and find the optimal system of subgroups of the group G having the rank s. This is done with the use of the fact that the subgroup classification problem reduces to classifying inequivalent subalgebras of the rank k of the Lie algebra g of the group G. This classification is performed within the action of the inner automorphism group of the algebra g.

(III) For each of the so obtained subgroups we construct the full set of functionally-independent invariants, which yields the invariant ansatz.

(IV) Inserting the above ansatz into the system of partial differential equations under study reduces it to the one having $n-s$ independent variables.

(V) We investigate the reduced system and construct its exact solutions. Each of them corresponds to the invariant solution of the initial system.

Symmetry properties of the overwhelming majority of physically significant differential equations (including the Maxwell and $SU(2)$ Yang-Mills equations) are well-known. The most important symmetry groups are those isomorphic to the Euclid, Galilei, Poincaré groups and their natural extensions (the Schrödinger and conformal groups). This fact was a motivation for investigation of the subgroup structure of these fundamental groups initiated by the paper by J.Patera, P.Winternitz and H.Zassenhaus [54]. They have suggested the general method for classifying continuous subgroups of Lie groups and illustrated its efficiency by re-deriving the known classification of inequivalent subgroups of the Poincaré group $P(1,3)$. Exploiting this method has enabled to get the full description of continuous subgroups of a number of important symmetry groups arising in theoretical and mathematical physics, including the Euclid, Galilei, Poincaré, Schrödinger and conformal groups (see, e.g., [30] and the references therein).

Thus to get the complete solution of the problem of symmetry reduction within the framework of the above formulated algorithm we need to be able to perform the remaining steps (III)–(V). However, solving these problems for a system of partial differential equations requires enormous amount of computations, and what is more, these computations cannot be fully automatized with the aid of symbolic computation routines. On the other hand, it is possible to simplify drastically the computations, if one notes that for the majority of physically important realizations of the Euclid, Galilei, Poincaré groups and of their extensions the corresponding invariant solutions admit linear representation. It was this very idea that had enabled constructing broad classes of invariant solutions of a number of nonlinear spinor equations [31]–[33].

In the sequel, we will concentrate on the case of the 15-parameter conformal group $C(1,3)$, admitted both by the Maxwell and $SU(2)$ Yang-Mills equations. We emphasize that the same reasoning applies directly to the case of the 11-parameter Schródinger group $Sch(1,3)$, which is the analogue of the conformal group in the non-relativistic physics. The group $C(1,3)$ acts in the open domain $M \subset R^{1,3} \times R^q$ of the four-dimensional Minkowski space-time of the independent variables $x_0, \mathbf{x} = (x_1, x_2, x_3)$ and of the q-dimensional space of dependent variables $\mathbf{u} = \mathbf{u}(x_0, \mathbf{x}),\ \mathbf{u} = (u^1, u^2, \ldots, u^q)$.

The Lie algebra $c(1,3)$ of the conformal group $C(1,3)$ is spanned by the generators of the translation $P_\mu,\ (\mu = 0,1,2,3)$, rotation $J_{ab},\ (a,b = 1,2,3,\ a<b)$, Lorentz rotation $J_{0a},\ (a = 1,2,3)$, dilation D and conformal K_μ, $(\mu = 0,1,2,3)$, transformations. The basis elements of $c(1,3)$ satisfy the

following commutation relations:

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha, \\
&[J_{\mu\nu}, J_{\alpha\beta}] = g_{\mu\beta} J_{\nu\alpha} + g_{\nu\alpha} J_{\mu\beta} - g_{\mu\alpha} J_{\nu\beta} - g_{\nu\beta} J_{\mu\alpha}, && (2.8)\\
&[P_\mu, D] = P_\mu, \quad [J_{\mu\nu}, D] = 0, && (2.9)\\
&[K_\mu, J_{\alpha\beta}] = g_{\mu\alpha} K_\beta - g_{\mu\beta} K_\alpha, \ [D, K_\mu] = K_\mu, \\
&[K_\mu, K_\nu] = 0, \ [P_\mu, K_\nu] = 2(g_{\mu\nu} D - J_{\mu\nu}). && (2.10)
\end{aligned}
$$

Here $\mu, \nu, \alpha, \beta = 0, 1, 2, 3$ and $g_{\mu\nu}$ is the metric tensor of the Minkowski space-time $R^{1,3}$, i.e.,

$$
g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0; \\ -1, & \mu = \nu = 1, 2, 3; \\ 0, & \mu \neq \nu. \end{cases}
$$

The group $C(1,3)$ contains the following important subgroups:

1) The Poincaré group $P(1,3)$, whose Lie algebra $p(1,3)$ is spanned by the operators $P_\mu, J_{\mu\nu}$, $(\mu, \nu = 0, 1, 2, 3)$ satisfying commutation relations (2.8);

2) the extended Poincaré group $\tilde{P}(1,3)$, whose Lie algebra $\tilde{p}(1,3)$ is spanned by the operators $P_\mu, J_{\mu\nu}, D$, $(\mu, \nu = 0, 1, 2, 3)$ satisfying commutation relations (2.8), (2.9).

Analysis of the symmetry groups of the equations of the relativistic physics shows that for the majority of them the generators of the Poincaré, extended Poincaré and conformal groups can be represented in the following form (see, e.g., [19, 21, 33, 48]):

$$
\begin{aligned}
P_\mu &= \partial_{x_\mu}, \\
J_{\mu\nu} &= x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu} - (S_{\mu\nu} \mathbf{u} \cdot \partial_{\mathbf{u}}), \\
D &= x_\mu \partial_{x_\mu} - k(E\mathbf{u} \cdot \partial_{\mathbf{u}}), \\
K_0 &= 2x_0 D - (x_\nu x^\nu) \partial_{x_0} - 2x_a (S_{0a} \mathbf{u} \cdot \partial_{\mathbf{u}}), \\
K_1 &= -2x_1 D - (x_\nu x^\nu) \partial_{x_1} + 2x_0 (S_{01} \mathbf{u} \cdot \partial_{\mathbf{u}}) \\
&\quad -2x_2 (S_{12} \mathbf{u} \cdot \partial_{\mathbf{u}}) - 2x_3 (S_{13} \mathbf{u} \cdot \partial_{\mathbf{u}}), \qquad (2.11)\\
K_2 &= -2x_2 D - (x_\nu x^\nu) \partial_{x_2} + 2x_0 (S_{02} \mathbf{u} \cdot \partial_{\mathbf{u}}) \\
&\quad +2x_1 (S_{12} \mathbf{u} \cdot \partial_{\mathbf{u}}) - 2x_3 (S_{23} \mathbf{u} \cdot \partial_{\mathbf{u}}), \\
K_3 &= -2x_3 D - (x_\nu x^\nu) \partial_{x_3} + 2x_0 (S_{03} \mathbf{u} \cdot \partial_{\mathbf{u}}) \\
&\quad +2x_1 (S_{13} \mathbf{u} \cdot \partial_{\mathbf{u}}) + 2x_2 (S_{23} \mathbf{u} \cdot \partial_{\mathbf{u}}).
\end{aligned}
$$

In formulae (2.11) $S_{\mu\nu}$ are constant $q \times q$ matrices, that realize a representation of the Lie algebra $o(1,3)$ of the pseudo-orthogonal group $O(1,3)$ and satisfy the commutation relations

$$
[S_{\mu\nu}, S_{\alpha\beta}] = g_{\mu\beta} S_{\nu\alpha} + g_{\nu\alpha} S_{\mu\beta} - g_{\mu\alpha} S_{\nu\beta} - g_{\nu\beta} S_{\mu\alpha}, \qquad (2.12)
$$

$\mu, \nu, \alpha, \beta = 0, 1, 2, 3$; $g_{\mu\nu}$ is the metric tensor of the Minkowski space $R^{1,3}$; E is the unit $q \times q$-matrix; $\mathbf{u} = (u^1, u^2, \dots, u^q)^T$; $\partial_{\mathbf{u}} = (\partial_{u^1}, \partial_{u^2}, \dots, \partial_{u^q})^T$; the symbol $(* \cdot *)$ stands for the scalar product in the vector space R^q. We remind that the repeated indices imply summation over the corresponding interval and raising and lowering the indices is carried out with the help of the metric $g_{\mu\nu}$. What is more, k is some fixed real number called the conformal degree of the group $c(1, 3)$.

It follows from relations (2.11) that the basis elements of the Lie algebra $c(1, 3)$ have the form (2.2), where the functions ξ^i_a depend on $\mathbf{x} \in X = R^p$ only and the functions η^a_j are linear in $\mathbf{u}$. We will prove that owing to these properties of the basis elements of $c(1, 3)$ the ansatzes invariant under subalgebras of the algebra (2.11) admit linear representation.

Let a local transformation group G act projectively in M, and let $g = \langle X_1, \dots, X_n \rangle$ be its Lie algebra spanned by the infinitesimal operators of the form

$$X_a = \xi^i_a(\mathbf{x})\partial_{x_i} + \rho^a_{jk}(\mathbf{x})u^k\partial_{u^j}, \tag{2.13}$$

where $a = 1, \dots, n,\ i = 1, \dots, p,\ j, k = 1, \dots, q$.

According to what was said above, the group G has the two types of invariants. The first set of invariants is formed by $p - s$ (where s is the rank of the group G) functionally independent invariants

$$\mathbf{w} = \mathbf{w}(\mathbf{x}),\ \mathbf{w} = (\omega^1, \dots, \omega^{p-s}). \tag{2.14}$$

The second set is formed by q invariants

$$\mathbf{h} = \mathbf{h}(\mathbf{x}, \mathbf{u}),\ \mathbf{h} = (h^1, \dots, h^q). \tag{2.15}$$

And what is more, the functions $\mathbf{w}$ and $\mathbf{h}$ are invariants of the group G if and only if they are, respectively, solutions of the following systems of partial differential equations:

$$\xi^i_a(\mathbf{x})\frac{\partial \omega^b}{\partial x_i} = 0, \tag{2.16}$$

$$\xi^i_a(\mathbf{x})\frac{\partial h^l}{\partial x_i} + \rho^a_{jk}(\mathbf{x})u^k\frac{\partial h^l}{\partial u^j} = 0. \tag{2.17}$$

In (2.16), (2.17) the indices take the following values, $a = 1, \dots, n$, $b = 1, \dots, p - s$, $i = 1, \dots, p$, $j, k, l = 1, \dots, q$.

Generically, a G-invariant ansatz has the form (2.6), where $\mathbf{v} \equiv \mathbf{h}$. However, provided the infinitesimal operators of the group G are of the form (2.13), G-invariant ansatz for the vector field $\mathbf{u}$ can be represented in the linear form [33]

$$\mathbf{u} = \Lambda(\mathbf{x})\mathbf{h}(\mathbf{w}), \tag{2.18}$$

where $\Lambda(\mathbf{x})$ is some $q \times q$ matrix non-singular in $\Omega \subset M$, $\mathbf{u} = (u^1, \dots, u^q)^T$, $\mathbf{h} = (h^1, \dots, h^q)^T$.

The matrix $\Lambda(\mathbf{x})$ from (2.18) is obtained by integrating the system of partial differential equations to be derived below.

Lema 2.1 *Let a G-invariant ansatz be of the form (2.18). Then there is $q \times q$-matrix $H(\mathbf{x}) = \Lambda^{-1}(\mathbf{x})$ non-singular in Ω satisfying the matrix partial differential equation*

$$\xi_a^i(\mathbf{x}) \frac{\partial H(\mathbf{x})}{\partial x_i} + H(\mathbf{x}) \Gamma_a(\mathbf{x}) = 0, \tag{2.19}$$

where $\Gamma_a(\mathbf{x})$ is the $q \times q$ matrices, whose (i,j)th entry reads as $\rho_{ij}^a(\mathbf{x})$, $i, j = 1, \dots, q$.

Proof. Provided a G-invariant ansatz is of the form (2.18), the relation

$$\mathbf{h} = H(\mathbf{x}) \mathbf{u}$$

with $H(\mathbf{x}) = \Lambda^{-1}(\mathbf{x})$ holds. So, the second set of invariants (2.15) of the group G consists of the functions, which are linear in u^j and, consequently, can be represented in the form

$$h^b = h_{bl}(\mathbf{x}) u^l, \quad b, l = 1, \dots, q.$$

The function h^b is the invariant of the group G, if and only if, it satisfies equation (2.17)

$$\xi_a^i(\mathbf{x}) \frac{\partial h_{bl}(\mathbf{x})}{\partial x_i} u^l + \rho_{jl}^a(\mathbf{x}) u^l h_{bj}(\mathbf{x}) = 0.$$

Splitting this relation by u^l yields that the system of partial differential equations

$$\xi_a^i(\mathbf{x}) \frac{\partial h_{bl}(\mathbf{x})}{\partial x_i} + h_{bj}(\mathbf{x}) \rho_{jl}^a(\mathbf{x}) = 0, \tag{2.20}$$

holds for all the values of b, l. The indices in (2.20) take the following values, $a = 1, \dots, n,\ i = 1, \dots, p,\ b, j, l = 1, \dots, q$.

It is readily seen that the second term of the left-hand side of equation (2.20) is the (b, l)th entry of the matrix $H(\mathbf{x}) \Gamma_a(\mathbf{x})$, $(a = 1, \dots, n)$. Hence it follows that the matrix $H(\mathbf{x})$ satisfies equation (2.19). The lemma is proved.

Below, we list the forms of the matrices Γ_a for the basis operators of the algebra $c(1,3)$

- matrices Γ_a, $a = 1, 2, 3, 4$ corresponding to the operators P_μ, $(\mu = 0, 1, 2, 3)$ are zero $q \times q$ matrices;

- matrices $\Gamma_a,\ a=1,\ldots,6$ corresponding to the operators $J_{\mu\nu},\ (\mu,\nu=0,1,2,3)$ are equal to $-S_{\mu\nu}$, where $S_{\mu\nu}$ are constant $q\times q$ matrices realizing a representation of the algebra $o(1,3)$ and satisfying commutation relations (2.12);

- the matrix Γ_1 corresponding to the dilation operator D reads as $-kE$, where k is the conformal degree of the algebra $c(1,3)$ and E is the unit $q\times q$ matrix;

- matrices $\Gamma_a,\ a=1,2,3,4$ corresponding to the operators $K_\mu,\ (\mu=0,1,2,3)$ are given by the following formulae:

$$\begin{aligned}
\Gamma_1 &= -2x_0kE-2x_1S_{01}-x_2S_{02}-x_3S_{03},\\
\Gamma_2 &= 2x_1kE+2x_0S_{01}-2x_2S_{12}-2x_3S_{13},\\
\Gamma_3 &= 2x_2kE+2x_0S_{02}+2x_1S_{12}-2x_3S_{23},\\
\Gamma_4 &= 2x_3kE+2x_0S_{03}+2x_1S_{13}+2x_2S_{23}.
\end{aligned}$$

With the explicit forms of the matrices Γ_a in hand we can determine the structure of the matrices $H=\Lambda^{-1}$ for ansatz (2.18) invariant under a subalgebra g of the conformal algebra $c(1,3)$.

If $g\in p(1,3)=\langle P_\mu, J_{\mu\nu}|\ \mu,\nu=0,1,2,3\rangle$, then the corresponding matrices Γ_a are linear combinations of the matrices $S_{\mu\nu}$. Hence it follows that the matrix H can be looked in the form

$$H=\tilde{H}=\prod_{\mu<\nu}\exp(\theta_{\mu\nu}S_{\mu\nu}), \tag{2.21}$$

where $\theta_{\mu\nu}=\theta_{\mu\nu}(x_0,\mathbf{x})$ are arbitrary smooth functions defined in $\tilde{\Omega}\subset R^{1,3}$.

Next, if g is a subalgebra of the conformal algebra $c(1,3)$ with a non-zero projection on the vector space spanned by the operators D, K_0, K_1, K_2, K_3, then the corresponding matrices Γ_a are linear combinations of the matrices E and $S_{\mu\nu}$. That is why, the matrix H should be looked for in the more general form

$$H=\exp(\theta E)\tilde{H}, \tag{2.22}$$

where $\theta=\theta(x_0,\mathbf{x})$ is an arbitrary smooth function defined in $\tilde{\Omega}$ and $\tilde{H}$ is the matrix given in (2.21).

2.2 Subalgebras of the conformal algebra $c(1,3)$ of the rank 3

Now we turn to the problem of constructing conformally-invariant ansatzes that reduce systems of partial differential equations invariant under the group $C(1,3)$ to systems of ordinary differential equations.

As a second step of the algorithm of symmetry reduction formulated above, we have to describe the optimal system of subalgebras of the algebra $c(1,3)$ of the rank $s=3$. Indeed, the initial system has $p=4$ independent variables. It has to be reduced to a system of differential equations in $4-s=1$ independent variables, so that $s=3$.

Classification of inequivalent subalgebras of the algebras $p(1,3)$, $\tilde{p}(1,3)$, $c(1,3)$ within actions of different automorphism groups (including the groups $P(1,3)$, $\tilde{P}(1,3)$ and $C(1,3)$) is already available (see, e.g., [30]). Since we will concentrate in the sequel on conformally-invariant systems, it is natural to restrict our considerations to the classification of subalgebras of $c(1,3)$ that are inequivalent within the action of the conformal group $C(1,3)$.

In order to get the full lists of the subalgebras in question we have to check that relation (2.3) with $s=3$ holds for each element of the lists of inequivalent subalgebras of the algebras $p(1,3), \tilde{p}(1,3), c(1,3)$ given in [30]. Evidently, we can restrict our considerations to subalgebras having the dimension not less than 3.

Let $c(1,3)$ be the conformal algebra having the basis operators (2.11) and $c^{(1)}(1,3)$ be the conformal algebra spanned by the operators

$$\begin{aligned} &P^{(1)}_\mu = \partial_{x_\mu}, \quad J^{(1)}_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \quad D^{(1)} = x_\mu \partial_{x_\mu}, \\ &K^{(1)} = 2x^\mu D^{(1)} - (x_\nu x^\nu)\partial_{x_\mu}, \end{aligned} \tag{2.23}$$

where $\mu, \nu = 0,1,2,3$.

Note that the conformal group $C(1,3)$ generated by the infinitesimal operators (2.23) acts in the space of independent variables $R^{1,3}$ only. That is why, the basis operators of the algebra $c^{(1)}(1,3)$ act in the space of dependent variables R^q as zero operators.

Lema 2.2 *Let L be a subalgebra of the algebra $c(1,3)$ of the rank s and let $s^{(1)}$ be the rank of the projection of L on $c^{(1)}(1,3)$. Then from the equality $s=s^{(1)}$ it follows that* $\dim L = s$.

Proof. Suppose that the reverse assertion holds, namely, that $L \neq s$. As $L \geq s$, hence it follows that $L > s$. Choose the basis elements $X_1, \ldots, X_m$ of the algebra L so that

- the rank of the matrix M, whose entries are projections of the operators $X_1, \ldots, X_m$ on $c^{(1)}(1,3)$, is equal to s, and
- the linear space spanned by the operators $X_1, \ldots, X_m$ contains $L \cap \langle P_0, P_1, P_2, P_3 \rangle$.

We denote as S_0 the point $(x^0_0, \mathbf{x}^0) \in \tilde{\Omega}$ in which the rank of the matrix M equals to s. Let the vector fields X_i be equal to $X^0_1, \ldots, X^0_s$, $X^0_{s+1}, \ldots X^0_m$ in S_0. Then there are constants $\alpha_1, \ldots, \alpha_s$, such that the vector field $\alpha_1 X^0_1 +$

$\ldots + \alpha_s X_s^0 + X_{s+1}^0$ restricted to the space of dependent variables $U = R^q$ is a non-zero operator. Indeed, if this operator vanishes identically on R^q for any choice of $\alpha_1, \ldots, \alpha_s$, then the vector fields $X_1^0, \ldots, X_s^0, X_{s+1}^0$ belong to the vector space $\langle P_0, P_1, P_2, P_3 \rangle$ and this fact contradicts to the assumptions that $\dim L > s,\ \mathrm{rank}\, L = s$. Consequently, the matrix formed by the coefficients of the vector fields $X^1, \ldots, X_s, \alpha_1 X_1 + \ldots + \alpha_s X_s + X_{s+1}$, has a non-zero minor of the order $s+1$ in some point $(x_0^0, \mathbf{x}^0, \mathbf{u}^0)$ (the first four coordinates are same as those of the point S_0). This contradicts to the assumption that $\dim L > s$. Hence we conclude that $\dim L = s$. The lemma is proved.

It follows from the above lemma that the validity of the relation (2.3) with $s = 3$ should be ascertained only for the three-dimensional subalgebras of the algebras $p(1,3), \tilde{p}(1,3), c(1,3)$ given in [30]. And what is more, we can restrict our considerations to checking the first condition from (2.3).

Consider the subalgebras of the algebra $p(1,3)$, whose basis operators are of the form (2.11). Among the three-dimensional subalgebras of the algebra $p(1,3)$ listed in [30] there are only five subalgebras $\langle G_1,\ P_0 + P_3,\ P_1 \rangle$, $\langle J_{12}, P_1, P_2 \rangle$, $\langle J_{03}, P_0, P_3 \rangle$, $\langle J_{12}, J_{13}, J_{23} \rangle$, $\langle J_{01}, J_{02}, J_{12} \rangle$, that do not respect the first condition (2.3). These subalgebras give rise to the so called partially invariant solutions (see, e.g., [19]). Partially invariant solutions cannot be handled in a generic way, they should always be considered within the context of a specific system of partial differential equation to be reduced. We exclude the partially invariant solutions from the further considerations. The remaining inequivalent subalgebras are listed in the assertion below.

Assertion 2.1 *The list of subalgebras of the algebra $p(1,3)$ of the rank 3, defined within the action of the inner automorphism group of the algebra $c(1,3)$, is exhausted by the following subalgebras:*

$$
\begin{aligned}
L_1 &= \langle P_0, P_1, P_2 \rangle; \quad L_2 = \langle P_1, P_2, P_3 \rangle;\\
L_3 &= \langle M, P_1, P_2 \rangle; \quad L_4 = \langle J_{03} + \alpha J_{12}, P_1, P_2 \rangle;\\
L_5 &= \langle J_{03}, M, P_1 \rangle; \quad L_6 = \langle J_{03} + P_1, P_0, P_3 \rangle;\\
L_7 &= \langle J_{03} + P_1, M, P_2 \rangle; \quad L_8 = \langle J_{12} + \alpha J_{03}, P_0, P_3 \rangle;\\
L_9 &= \langle J_{12} + P_0, P_1, P_2 \rangle; \quad L_{10}^j = \langle J_{12} + (-1)^j P_3, P_1, P_2 \rangle;\\
L_{11}^j &= \langle J_{12} + (-1)^j 2T, P_1, P_2 \rangle; \quad L_{12} = \langle G_1, M, P_2 + \alpha P_1 \rangle;\\
L_{13}^j &= \langle G_1 + (-1)^j P_2, M, P_1 \rangle; \quad L_{14} = \langle G_1 + 2T, M, P_2 \rangle;\\
L_{15} &= \langle G_1 + 2T, M, P_1 + \alpha P_2 \rangle; \quad L_{16} = \langle J_{12}, J_{03}, M \rangle;\\
L_{17}^j &= \langle G_1^j, G_2^j, M \rangle; \quad L_{18} = \langle J_{03}, G_1, P_2 \rangle;\\
L_{19} &= \langle G_1, J_{03}, M \rangle; \quad L_{20} = \langle G_1, J_{03} + P_2, M \rangle;\\
L_{21} &= \langle G_1, J_{03} + P_1 + \alpha P_2, M \rangle; \quad L_{22} = \langle G_1, G_2, J_{03} + \alpha J_{12} \rangle,
\end{aligned}
$$

where $\alpha \in \mathbf{R}$; $M = P_0 + P_3,\ T = \frac{1}{2}(P_0 - P_3),\ G_a = J_{0a} - J_{a3},\ (a = 1, 2)$; $G_1^j = G_1 + (-1)^j P_2,\ G_2^j = G_2 - (-1)^j P_1 + \alpha P_2$; *j=1,2.*

In the same way, we handle the three-dimensional subalgebras of the algebras $\tilde{p}(1,3)$ and $c(1,3)$. We have skipped from the list of subalgebras of the algebra $\tilde{p}(1,3)$ those conjugate to subalgebras of $p(1,3)$. Furthermore, we have skipped from the list of subalgebras of the conformal algebra those conjugate to subalgebras of the algebra $\tilde{p}(1,3)$. The results obtained are presented in the two assertions below.

Assertion 2.2 *The list of subalgebras of the algebra $\tilde{p}(1,3)$ of the rank 3, defined within the action of the inner automorphism group of the algebra $c(1,3)$, is exhausted by the subalgebras given in Assertion 2.1 and by the following subalgebras:*

$$\begin{aligned}
F_1 &= \langle D, P_0, P_3\rangle; \quad F_2 = \langle J_{12} + \alpha D, P_0, P_3\rangle;\\
F_3 &= \langle J_{12}, D, P_0\rangle; \quad F_4 = \langle J_{12}, D, P_3\rangle;\\
F_5 &= \langle J_{03} + \alpha D, P_0, P_3\rangle; \quad F_6 = \langle J_{03} + \alpha D, P_1, P_2\rangle;\\
F_7 &= \langle J_{03} + \alpha D, M, P_1\rangle \ (a \neq 0);\\
F_8 &= \langle J_{03} + D + (-1)^j 2T, P_1, P_2\rangle;\\
F_9 &= \langle J_{03} + D + (-1)^j 2T, M, P_1\rangle; \quad F_{10} = \langle J_{03}, D, P_1\rangle;\\
F_{11} &= \langle J_{03}, D, M\rangle; \quad F_{12} = \langle J_{12} + \alpha J_{03} + \beta D, P_0, P_3\rangle \ (\alpha \neq 0);\\
F_{13} &= \langle J_{12} + \alpha J_{03} + \beta D, P_1, P_2\rangle \ (\alpha \neq 0);\\
F_{14} &= \langle J_{12} + \alpha(J_{03} + D + 2T), P_1, P_2\rangle \ (\alpha \neq 0);\\
F_{15} &= \langle J_{12} + \alpha J_{03}, D, M\rangle \ (\alpha \neq 0);\\
F_{16} &= \langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle \ (0 \leq |\alpha| \leq 1, \ \beta \geq 0, |\alpha| + |\beta| \neq 0);\\
F_{17} &= \langle J_{03} + D + (-1)^j 2T, J_{12} + 2\alpha T, M\rangle \ (\alpha \in R);\\
F_{18} &= \langle J_{03} + D, J_{12} + (-1)^j 2T, M\rangle; \quad F_{19} = \langle J_{03}, J_{12}, D\rangle;\\
F_{20} &= \langle G_1, J_{03} + \alpha D, P_2\rangle \ (0 < |\alpha| \leq 1);\\
F_{21} &= \langle J_{03} + D, G_1 + (-1)^j P_2, M\rangle;\\
F_{22} &= \langle J_{03} - D + (-1)^j M, G_1, P_2\rangle;\\
F_{23} &= \langle J_{03} + 2D, G_1 + (-1)^j 2T, M\rangle;\\
F_{24} &= \langle J_{03} + 2D, G_1 + (-1)^j 2T, P_2\rangle.
\end{aligned}$$

Here $M = P_0 + P_3$, $G_1 = J_{01} - J_{13}$, $T = \frac{1}{2}(P_0 - P_3)$, the parameters α, β are positive (if otherwise is not indicated); $j = 1, 2$.

Assertion 2.3 *The list of subalgebras of the algebra $c(1,3)$ of the rank 3, defined within the action of the inner automorphism group of the algebra $c(1,3)$, is exhausted by the subalgebras of the algebras $p(1,3), \tilde{p}(1,3)$ given in Assertions 2.1 and 2.2 and by the following subalgebras:*

$$\begin{aligned}
C_1 &= \langle S + T + J_{12}, G_1 + P_2, M\rangle;\\
C_2 &= \langle S + T + J_{12} + G_1 + P_2, G_2 - P_1, M\rangle;
\end{aligned}$$

$$
\begin{aligned}
C_3 &= \langle J_{12}, S+T, M\rangle; \quad C_4 = \langle S+T, Z, M\rangle;\\
C_5 &= \langle S+T+\alpha J_{12}, Z, M\rangle \ (\alpha \neq 0);\\
C_6 &= \langle S+T+J_{12}+\alpha Z, G_1+P_2, M\rangle \ (\alpha \neq 0);\\
C_7 &= \langle S+T+J_{12}, Z, G_1+P_2\rangle;\\
C_8 &= \langle S+T+\beta Z, J_{12}+\alpha Z, M\rangle \quad (\alpha, \beta \in R, |\alpha|+|\beta| \neq 0);\\
C_9 &= \langle J_{12}, S+T, Z\rangle; \quad C_{10} = \langle D-J_{03}, S, T\rangle;\\
C_{11} &= \langle P_2+K_2+\sqrt{3}(P_1+K_1)+K_0-P_0,\\
&\quad -D+J_{02}-\sqrt{3}J_{01}, P_0+K_0-2(K_2-P_2)\rangle;\\
C_{12} &= \langle P_0+K_0\rangle \oplus \langle J_{12}, K_3-P_3\rangle;\\
C_{13} &= \langle 2J_{12}+K_3-P_3, 2J_{13}-K_2+P_2, 2J_{23}+K_1-P_1\rangle;\\
C_{14} &= \langle P_1+K_1+2J_{03}, P_2+K_2+K_0-P_0, 2J_{12}+K_3-P_3\rangle,
\end{aligned}
$$

where $M = P_0 + P_3$, $G_{0a} = J_{0a} - J_{a3}$, $(a = 1, 2)$, $Z = J_{03} + D, S = \frac{1}{2}(K_0 + K_3)$, $T = \frac{1}{2}(P_0 - P_3)$.

Remark. While classifying subalgebras of the extended Poincaré algebra $\tilde{p}(1,3)$, the discrete equivalence transformations Φ_1, Φ_2, Φ_3, that leave the algebra $\tilde{p}(1,3)$ invariant, were exploited in [30]. The result of the action of these groups on the operators of the algebra $\tilde{p}(1,3)$ is given in Table 2.1. That is why, we have completed the list of subalgebras of the algebras $p(1,3), \tilde{p}(1,3)$ obtained in [30] by the subalgebras obtainable by acting on these subalgebras with the discrete transformation groups Φ_1, Φ_2, Φ_3.

Table 2.1.

Operators	Action on $\tilde{p}(1,3)$		
	Φ_1	Φ_2	Φ_3
P_0	$-P_0$	P_0	$-P_0$
P_1	$-P_1$	$-P_1$	P_1
$P_a \quad (a = 2, 3)$	$-P_a$	P_a	$-P_a$
J_{03}	J_{03}	J_{03}	J_{03}
J_{12}	J_{12}	$-J_{12}$	$-J_{12}$
G_1	G_1	$-G_1$	$-G_1$
G_2	G_2	G_2	G_2
M	$-M$	M	$-M$
T	$-T$	T	$-T$
D	D	D	D

2.3 Construction of conformally-invariant ansatzes

Now we turn to constructing $C(1,3)$-invariant ansatzes that reduce conformally-invariant systems of partial differential equations to systems of ordinary

differential equations. To this end, we use the lists of subalgebras of the algebra $c(1,3)$ given in Assertions 2.1–2.3. Note that all the subsequent computations are performed under supposition that the basis operators of $c(1,3)$ are of the form (2.11).

As shown in Subsection 2.1, the ansatzes in question can be looked for in the linear form (2.18), matrices $H=\Lambda^{-1}$ being searched for in the form (2.22). According to Lemma 2.1, the matrix H has to satisfy equations (2.19), whose coefficients are defined uniquely by the choice of a subalgebra of the conformal algebra of the rank 3. So that, the problem of complete description of conformally-invariant ansatzes reduces to solving system of partial differential equations (2.16), (2.19) for each of the subalgebras of the conformal algebra, which requires the huge amount of computations. The calculations simplify essentially, if we take into account the general structure of the subalgebras listed in Assertions 2.1–2.3.

For the further convenience, we will use the following basis of the algebra $o(1,3)$: $S_{03},\ S_{12},\ H_a,\ \tilde{H}_a,\ (a=1,2)$, where $H_a=S_{0a}-S_{a3},\ \tilde{H}_a=S_{0a}+S_{a3},\ (a=1,2)$. It is not difficult to check that these matrices satisfy the commutation relations

$$\begin{aligned}
&[S_{03},S_{12}]=[H_1,H_2]=[\tilde{H}_1,\tilde{H}_2]=0,\\
&[H_a,S_{03}]=H_a,[\tilde{H}_a,S_{03}]=-\tilde{H}_a,\ (a=1,2),\\
&[H_1,S_{12}]=-H_2,[H_2,S_{12}]=H_1,\\
&[\tilde{H}_1,S_{12}]=-\tilde{H}_2,\ [\tilde{H}_2,S_{12}]=\tilde{H}_1,\\
&[H_1,\tilde{H}_1]=[H_2,\tilde{H}_2]=-2S_{03},\\
&[\tilde{H}_2,H_1]=[H_2,\tilde{H}_1]=2S_{12}.
\end{aligned}\tag{2.24}$$

In particular, relations (2.24) imply that the matrices H_1,H_2,S_{12},S_{03} and $\tilde{H}_1,\tilde{H}_2,S_{12},S_{03}$ realize two matrix representations of the Euclid algebra $\tilde{e}(2)$ (here the matrix S_{03} is identified with the dilation generator and the matrices H_1,H_2 and $\tilde{H}_1,\tilde{H}_2$ are identified with the translation generators). Furthermore, as E is the unit matrix, it commutes with all the basis elements of $o(1,3)$, namely,

$$[E,S_{12}]=[E,S_{03}]=[E,H_a]=[E,\tilde{H}_a]=0,\tag{2.25}$$

where $a=1,2$.

Analyzing the structure of the basis elements of the subalgebras of the conformal algebra given in Assertions 2.1–2.3 we see that the corresponding matrices Γ_a are most conveniently represented in terms of the matrices S_{03}, S_{12}, H_a, $\tilde{H}_a$, $(a=1,2)$. Hence we conclude that the matrix $H=H(x_0,\mathbf{x})=\Lambda^{-1}(x_0,\mathbf{x})$ can be looked for in the form

$$\begin{aligned}
H\ =\ &\exp\{(-\ln\theta)E\}\exp(\theta_0S_{03})\exp(-\theta_3S_{12})\exp(-2\theta_1H_1)\\
&\times\exp(-2\theta_2H_2)\exp(-2\theta_4\tilde{H}_1)\exp(-2\theta_5\tilde{H}_2),
\end{aligned}\tag{2.26}$$

where $\theta = \theta(x_0, \mathbf{x}),\ \theta_0 = \theta_0(x_0, \mathbf{x}), \theta_m = \theta_m(x_0, \mathbf{x}),\ (m = 1, 2, \ldots, 5)$ are arbitrary smooth functions defined in an open domain $\tilde{\Omega} \subset R^{1,3}$ of the Minkowski space of the independent variables $x_0, \mathbf{x} = (x_1, x_2, x_3)$.

Let $L = \langle X_a | a = 1, 2, 3\rangle$ be a subalgebra of the algebra $c(1, 3)$ of the rank 3. By assumption, the basis operators of L can be written in the following form:

$$X_a = \xi_a^\mu(x_0, \mathbf{x})\partial_{x_\mu} + (\tilde{\Gamma}_a \mathbf{u} \cdot \partial_{\mathbf{u}}),\ a = 1, 2, 3, \tag{2.27}$$

and what is more,

$$\tilde{\Gamma}_a = f^a E + f_0^a S_{03} + f_1^a H_1 + f_2^a H_2 + f_3^a S_{12} + f_4^a \tilde{H}_1 + f_5^a \tilde{H}_2,\ (a = 1, 2, 3), \tag{2.28}$$

where $f^a = f^a(x_0, \mathbf{x}),\ f_0^a = f_0^a(x_0, \mathbf{x}), f_m^a = f_m^a(x_0, \mathbf{x}),\ (m = 1, \ldots, 5)$ are some fixed smooth functions. In particular, if the operator X_a is a linear combination of the translation generators, then $\Gamma_a = 0$, and therefore, $f^a = f_0^a = f_m^a = 0$ in (2.28).

Owing to Lemma 2.1, in order to construct ansatz (2.18) invariant under the subalgebra L, we have to solve systems (2.16), (2.19), which in the case under consideration read as

$$\xi_a^\mu \frac{\partial \omega}{\partial x_\mu} = 0, \tag{2.29}$$

$$\xi_a^\mu \frac{\partial H}{\partial x_\mu} + H\tilde{\Gamma}_a = 0, \tag{2.30}$$

where $a = 1, 2, 3,\ \mu = 0, 1, 2, 3$. The functions $\xi_a^\mu = \xi_a^\mu(x_0, \mathbf{x})$ and the variable matrices $\tilde{\Gamma}_a = \tilde{\Gamma}(x_0, \mathbf{x})$ are the coefficients of the basis operators of the subalgebra L (note that $\tilde{\Gamma}_a$ is of the form (2.28)). Matrix function H (2.26) and the scalar function $\omega = \omega(x_0, \mathbf{x})$ are to be determined while integrating (2.29), (2.30).

Next, we will prove a technical assertion to be used in the sequel for simplifying the form of system (2.30).

Lema 2.3 *Let H be of the form (2.26). Then the following identity holds true:*

$$\begin{aligned}
\xi_a^\mu \frac{\partial H}{\partial x_\mu} \quad = \quad & H\Big\{-\theta^{-1}\xi_a^\mu \frac{\partial \theta}{\partial x_\mu} E + \xi_a^\mu \frac{\partial \theta_0}{\partial x_\mu}[(1 + 8\theta_1\theta_4 + 8\theta_2\theta_5)S_{03} \\
& +8(\theta_1\theta_5 - \theta_2\theta_4)S_{12} + 2\theta_1 H_1 + 2\theta_2 H_2 - 2(\theta_4 + 4\theta_1\theta_4^2 + 8\theta_2\theta_4\theta_5 \\
& -4\theta_1\theta_5^2)\tilde{H}_1 - 2(\theta_5 + 4\theta_2\theta_5^2 + 8\theta_1\theta_4\theta_5 - 4\theta_2\theta_4^2)\tilde{H}_2] \\
& -\xi_a^\mu \frac{\partial \theta_3}{\partial x_\mu}[8(\theta_2\theta_4 - \theta_1\theta_5)S_{03} + (1 + 8\theta_1\theta_4 + 8\theta_2\theta_5)S_{12} \\
& +2\theta_2 H_1 - 2\theta_1 H_2 + 2(\theta_5 + 4\theta_2\theta_5^2 - 4\theta_2\theta_4^2 + 8\theta_1\theta_4\theta_5)\tilde{H}_1 \\
& -2(\theta_4 + 4\theta_1\theta_4^2 - 4\theta_1\theta_5^2 + 8\theta_2\theta_4\theta_5)\tilde{H}_2]
\end{aligned}$$

$$
\begin{aligned}
&-2\xi_a^\mu \frac{\partial\theta_1}{\partial x_\mu}[4\theta_4 S_{03} + 4\theta_5 S_{12} + H_1 + 4(\theta_5^2 - \theta_4^2)\tilde{H}_1 - 8\theta_4\theta_5\tilde{H}_2] \\
&-2\xi_a^\mu \frac{\partial\theta_2}{\partial x_\mu}[4\theta_5 S_{03} - 4\theta_4 S_{12} + H_2 - 8\theta_4\theta_5\tilde{H}_1 + 4(\theta_4^2 - \theta_5^2)\tilde{H}_2] \\
&-2\xi_a^\mu \frac{\partial\theta_4}{\partial x_\mu}\tilde{H}_1 - 2\xi_a^\mu \frac{\partial\theta_5}{\partial x_\mu}\tilde{H}_2]\Big\},
\end{aligned}
$$

where $a = 1, 2, 3,\ \ \mu = 0, 1, 2, 3.$

Proof. Acting by the linear differential operator $\xi_a^\mu \partial_{x_\mu}$ on matrix H (2.26) yields the equality, whose right-hand side can be decomposed into the sum of seven terms having the same structure

$$
\xi_a^\mu \frac{\partial H}{\partial x_\mu} = \sum_{i=1}^{7} D_i. \tag{2.31}
$$

As each of the terms D_i is handled in the same way, we give the calculation details for one of them, say, for

$$
D_4 = \exp\{(-\ln\theta)E\} \prod_{i=1}^{3} \Lambda_i(-2\xi_a^\mu \frac{\partial\theta_1}{\partial x_\mu} H_1) \prod_{j=4}^{6} \Lambda_j. \tag{2.32}
$$

Note that in (2.32) we use the following designations:

$$
\begin{aligned}
&\Lambda_1 = \exp(\theta_0 S_{03}), \quad \Lambda_2 = \exp(-\theta_3 S_{12}), \\
&\Lambda_3 = \exp(-2\theta_1 H_1), \quad \Lambda_4 = \exp(-2\theta_3 H_2), \\
&\Lambda_5 = \exp(-2\theta_4 \tilde{H}_1), \quad \Lambda_6 = \exp(-2\theta_5 \tilde{H}_2).
\end{aligned} \tag{2.33}
$$

Having multiplied the right-hand side of (2.32) by the matrix HH^{-1} on the left, we arrive at the equality

$$
D_4 = H\Big(-2\xi_a^\mu \frac{\partial\theta_1}{\partial x_\mu}\Big)\Lambda_6^{-1}\Lambda_5^{-1}\Lambda_4^{-1} H_1 \Lambda_4\Lambda_5\Lambda_6, \tag{2.34}
$$

where the matrices $\Lambda_4, \Lambda_5, \Lambda_6$ are given in (2.33).

To simplify the right-hand side of (2.34) we exploit the Campbell-Hausdorff formula

$$
\begin{aligned}
&\exp(\tau A) B \exp(-\tau A) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\{A, B\}^n, \\
&\{A, B\}^n = [A, \{A, B\}^{n-1}], \quad \{A, B\}^0 = B,
\end{aligned}
$$

that holds for arbitrary square matrices A, B.

With account of commutation relations (2.24), (2.25) we get

$$
\Lambda_4^{-1} H_1 \Lambda_1 = \exp(2\theta_2 H_2) H \exp(-2\theta_2 H_2) = H_1,
$$

whence

$$\Lambda_5^{-1}\Lambda_4^{-1}H\Lambda_4\Lambda_5 = \Lambda_5^{-1}H_1\Lambda_5 = \exp(2\theta_4\tilde{H}_1)H_1\exp(-2\theta_4\tilde{H}_1)$$
$$= H_1 + 4\theta_4 S_{03} - 4\theta_4^2\tilde{H}_1.$$

Consequently,

$$\Lambda_6^{-1}\Lambda_5^{-1}\Lambda_4^{-1}H_1\Lambda_4\Lambda_5\Lambda_6 = \exp(2\theta_5\tilde{H}_2)(H_1 + 4\theta_4 S_{03} - 4\theta_2^2\tilde{H}_1)$$
$$\times\exp(-2\theta_5\tilde{H}_2) = H_1 + 4\theta_4 S_{03} + 4\theta_5 S_{12} + 4(\theta_5^2 - \theta_4^2)\tilde{H}_1 - 8\theta_4\theta_5\tilde{H}_2.$$

Finally, we have

$$D_4 = H(-2\xi_a^\mu\frac{\partial\theta_2}{\partial x_\mu})[H_1 + 4\theta_4 S_{03} + 4\theta_5 S_{12} + 4(\theta_5^2 - \theta_4^2)\tilde{H}_1 - 8\theta_4\theta_5\tilde{H}_2].$$

The same reasonings, when applied to the remaining terms of the right-hand side of the equality (2.31), complete the proof of the lemma.

Assertion 2.4 *System (2.30) is equivalent to the system of partial differential equations for the functions $\theta, \theta_0, \theta_m$, $(m = 1, 2, \ldots, 5)$*

$$\begin{aligned}
\xi_a^\mu\frac{\partial\theta}{\partial x_\mu} &= f^a\theta,\\
\xi_a^\mu\frac{\partial\theta_0}{\partial x_\mu} &= 4(\theta_4 f_1^a + \theta_5 f_2^a) - f_0^a,\\
\xi_a^\mu\frac{\partial\theta_1}{\partial x_\mu} &= 4(\theta_1\theta_4 + \theta_2\theta_5)f_1^a\\
&\quad +4(\theta_1\theta_5 - \theta_2\theta_4)f_2^a - \theta_1 f_0^a - \theta_2 f_3^a + \frac{1}{2}f_1^a,\\
\xi_a^\mu\frac{\partial\theta_2}{\partial x_\mu} &= 4(\theta_2\theta_4 - \theta_1\theta_5)f_1^a\\
&\quad +4(\theta_2\theta_5 + \theta_1\theta_4)f_2^a - \theta_2 f_0^a + \theta_1 f_3^a + \frac{1}{2}f_2^a,\\
\xi_a^\mu\frac{\partial\theta_3}{\partial x_\mu} &= 4(\theta_4 f_2^a - \theta_5 f_1^a) + f_3^a, \qquad (2.35)\\
\xi_a^\mu\frac{\partial\theta_4}{\partial x_\mu} &= \theta_4 f_0^a - 2(\theta_4^2 - \theta_5^2)f_1^a - 4\theta_4\theta_5 f_2^a - \theta_5 f_3^a + \frac{1}{2}f_4^a,\\
\xi_a^\mu\frac{\partial\theta_5}{\partial x_\mu} &= \theta_5 f_0^a - 4\theta_4\theta_5 f_1^a + 2(\theta_4^2 - \theta_5^2)f_2^a + \theta_4 f_3^a + \frac{1}{2}f_5^a.
\end{aligned}$$

In (2.35) $\mu = 0, 1, 2, 3$; $a = 1, 2, 3$. The coefficients of linear differential operators $\xi_a^\mu\partial_{x_\mu}$ and the functions f^a, f_0^a, f_m^a, $(m = 1, 2, \ldots, 5)$ are defined by the coefficients of the basis operators of the subalgebra L of the algebra $c(1,3)$ of the rank 3.

Proof. Inserting the expression for $\xi_a^\mu \frac{\partial H}{\partial x_\mu}$ given in Lemma 2.3 into the left-hand side of (2.30) and multiplying the obtained equation by the inverse of the non-singular matrix H we arrive at the system of matrix equations, whose left-hand sides are the linear combinations of the linearly independent matrices $E, S_{01}, S_{12}, H_a, \tilde{H}_a,\ (a = 1, 2)$. Splitting the system obtained by these matrices, and taking into account the forms of the matrices $\tilde{\Gamma}_a$, and performing some simplifications yield system of equations (2.35). The assertion is proved.

Summarizing we conclude that the problem of constructing conformally-invariant ansatzes reduces to finding the fundamental solution of the system of linear partial differential equations (2.29) and particular solutions of first-order system of nonlinear partial differential equations (2.35).

The next subsections are devoted to constructing the ansatzes invariant under the subalgebras of the Poincaré, extended Poincaré and conformal algebras given in Assertions 2.1–2.3. The solution procedure is based on the above derived identities and, essentially, on Assertion 2.4.

2.3.1 $P(1,3)$-invariant ansatzes

Subalgebras listed in Assertion 2.1 give rise to $P(1,3)$- (Poincaré-) invariant ansatzes. Analysis of the structure of these subalgebras shows that we can put $\theta = 1, \theta_4 = \theta_5 = 0$ in formula (2.26) for the matrix H. What is more, the form of the basis elements of these subalgebras imply that in formulae (2.28), (2.34) $f^a = f_4^a = f_5^a = 0$, for all the values of $a = 1, 2, 3$. Owing to these facts, system (2.35) for the matrix H takes the form of twelve first-order partial differential equations for the functions $\theta_0, \theta_1, \theta_2, \theta_3$

$$\begin{aligned}
&\xi_a^\mu \frac{\partial \theta_0}{\partial x_\mu} = -f_0^a, \quad \xi_a^\mu \frac{\partial \theta_3}{\partial x_\mu} = f_3^a, \\
&\xi_a^\mu \frac{\partial \theta_1}{\partial x_\mu} = -\theta_1 f_0^a - \theta_2 f_3^a + \frac{1}{2} f_1^a, \\
&\xi_a^\mu \frac{\partial \theta_2}{\partial x_\mu} = -\theta_2 f_0^a + \theta_1 f_3^a + \frac{1}{2} f_2^a,
\end{aligned} \tag{2.36}$$

where $\mu = 0, 1, 2, 3; a = 1, 2, 3$.

We integrate system (2.29) for the case of the subalgebra $L_{22} = \langle G_1,\ G_2,\ J_{03} + \alpha J_{12} \rangle$, $(\alpha \in R)$ (all other cases are handled in a similar way).

System (2.29) for finding the function $\omega = \omega(x_0, \mathbf{x})$ reads as

$$\begin{aligned}
&G_1^{(1)}\omega = [(x_0 - x_3)\partial_{x_1} + x_1(\partial_{x_0} + \partial_{x_3})]\omega = 0, \\
&G_2^{(1)}\omega = [(x_0 - x_3)\partial_{x_2} + x_2(\partial_{x_0} + \partial_{x_3})]\omega = 0, \\
&(J_{03}^{(1)} + \alpha J_{12}^{(1)})\omega = [x_0\partial_{x_3} + x_3\partial_{x_0} + \alpha(x_2\partial_{x_1} - x_1\partial_{x_2})]\omega = 0, \quad \alpha \in R.
\end{aligned} \tag{2.37}$$

On making the change of variables

$$\begin{aligned} y_0 &= (x_0+x_3)(x_0-x_3), \quad y_1 = \sqrt{x_1^2+x_2^2},\\ y_2 &= \arctan\frac{x_2}{x_1}, \quad y_3 = x_0-x_3, \end{aligned}$$

reduces system (2.37) to the form

$$\begin{aligned} &y_1\frac{\partial\omega}{\partial y_1} + 2y_1^2\frac{\partial\omega}{\partial y_0} - \tan y_2\frac{\partial\omega}{\partial y_2} = 0,\\ &y_1\frac{\partial\omega}{\partial y_1} + 2y_1^2\frac{\partial\omega}{\partial y_0} - (\tan y_2)^{-1}\frac{\partial\omega}{\partial y_2} = 0,\\ &y_3\frac{\partial\omega}{\partial y_3} + \alpha\frac{\partial\omega}{\partial y_2} = 0. \end{aligned}$$

The fundamental solution of the above system reads as $\omega = y_0 - y_1^2$. Returning back to the initial variables, we get the fundamental solution of system (2.37), $\omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Next, taking into account the forms of the basis elements of the subalgebra L_{22}, we get the expressions for the functions f_μ^a, $(\mu = 0,1,2,3;\ a = 1,2,3)$

$$\begin{aligned} G_1 &: \ f_0^1 = f_2^1 = f_3^1 = 0, \quad f_1^1 = -1;\\ G_2 &: \ f_0^2 = f_1^2 = f_3^2 = 0, \quad f_2^2 = -1;\\ J_{03}+\alpha J_{12} &: \ f_0^3 = -1, \quad f_1^3 = f_2^3 = 0, \quad f_3^3 = -\alpha, \quad (\alpha\in R). \end{aligned}$$

So that, system (2.36) takes the form

$$\begin{aligned} &G_1^{(1)}\theta_0 = G_1^{(1)}\theta_2 = G_1^{(1)}\theta_3 = 0, \quad G_1^{(1)}\theta_1 = -\frac{1}{2},\\ &G_2^{(1)}\theta_0 = G_2^{(1)}\theta_1 = G_2^{(1)}\theta_3 = 0, \quad G_2^{(1)}\theta_2 = -\frac{1}{2},\\ &(J_{03}^{(1)} + \alpha J_{12}^{(1)})\theta_0 = 1, \quad (J_{03}^{(1)} + \alpha J_{12}^{(1)})\theta_3 = -\alpha,\\ &(J_{03}^{(1)} + \alpha J_{12}^{(1)})\theta_1 = \theta_1 + \alpha\theta_2, \quad (J_{03}^{(1)} + \alpha J_{12}^{(1)})\theta_2 = \theta_2 - \alpha\theta_1. \end{aligned} \tag{2.38}$$

As we have already mentioned, to construct the matrix H it suffices to find particular solutions of system (2.38). The system for determination of the function θ_0 reads as

$$\begin{aligned} &(x_0-x_3)\frac{\partial\theta_0}{\partial x_a} + x_a\Big(\frac{\partial\theta_0}{\partial x_0} + \frac{\partial\theta_0}{\partial x_3}\Big) = 0, \ (a=1,2),\\ &x_0\frac{\partial\theta_0}{\partial x_3} + x_3\frac{\partial\theta_0}{\partial x_0} + \alpha\Big(x_2\frac{\partial\theta_0}{\partial x_1} - x_1\frac{\partial\theta_0}{\partial x_2}\Big) = 1. \end{aligned} \tag{2.39}$$

We look for its particular solution of the form $\theta_0 = f(x_0 - x_3)$. By the direct check we become convinced of the fact that this function satisfies the first

two equations of system (2.39), and furthermore, the third one reduces to the ordinary differential equation

$$-\xi \frac{df}{d\xi} = 1, \quad \xi = x_0 - x_3,$$

whose solution reads as $f = -\ln|\xi|$.

Thus we can choose $\theta_0 = -\ln|x_0 - x_3|$. The first two equations for the function θ_3 coincide with those from (2.39) and the third equation

$$x_0 \frac{\partial \theta_3}{\partial x_3} + x_3 \frac{\partial \theta_3}{\partial x_0} + \alpha\Big(x_2 \frac{\partial \theta_3}{\partial x_1} - x_1 \frac{\partial \theta_3}{\partial x_2}\Big) = -\alpha$$

differs from the third equation from system (2.39) by the constant $-\alpha$ in the right-hand side. Owing to these remarks, we easily get the final form of the particular solution of (2.39)

$$\theta_3 = \alpha \ln|x_0 - x_3|.$$

According to (2.38) the system for finding the functions θ_1, θ_2 has the form

$$\begin{aligned}
&(x_0 - x_3)\frac{\partial \theta_1}{\partial x_1} + x_1\Big(\frac{\partial \theta_1}{\partial x_0} + \frac{\partial \theta_1}{\partial x_3}\Big) = -\frac{1}{2},\\
&(x_0 - x_3)\frac{\partial \theta_2}{\partial x_1} + x_1\Big(\frac{\partial \theta_2}{\partial x_0} + \frac{\partial \theta_2}{\partial x_3}\Big) = 0,\\
&(x_0 - x_3)\frac{\partial \theta_1}{\partial x_2} + x_2\Big(\frac{\partial \theta_1}{\partial x_0} + \frac{\partial \theta_2}{\partial x_3}\Big) = 0,\\
&(x_0 - x_3)\frac{\partial \theta_2}{\partial x_2} + x_2\Big(\frac{\partial \theta_2}{\partial x_0} + \frac{\partial \theta_2}{\partial x_3}\Big) = -\frac{1}{2},\\
&x_0 \frac{\partial \theta_1}{\partial x_3} + x_3 \frac{\partial \theta_1}{\partial x_0} - \alpha\Big(x_1 \frac{\partial \theta_1}{\partial x_2} - x_2 \frac{\partial \theta_1}{\partial x_1}\Big) = \theta_1 + \alpha\theta_2,\\
&x_0 \frac{\partial \theta_2}{\partial x_3} + x_3 \frac{\partial \theta_2}{\partial x_0} - \alpha\Big(x_1 \frac{\partial \theta_2}{\partial x_2} - x_2 \frac{\partial \theta_2}{\partial x_1}\Big) = \theta_2 - \alpha\theta_1.
\end{aligned} \tag{2.40}$$

We seek for its solutions of the form

$$\theta_1 = g(\xi, x_1), \quad \theta_2 = h(\xi, x_2), \quad \xi = x_0 - x_3. \tag{2.41}$$

Inserting functions (2.41) into system (2.40) reduces it to the form

$$\begin{aligned}
&\xi \frac{\partial g}{\partial x_1} = -\frac{1}{2}, \quad \xi \frac{\partial h}{\partial x_2} = -\frac{1}{2},\\
&-\xi \frac{\partial g}{\partial \xi} + \alpha x_2 \frac{\partial g}{\partial x_1} = g + \alpha h,\\
&-\xi \frac{\partial h}{\partial \xi} - \alpha x_1 \frac{\partial h}{\partial x_2} = h - \alpha g.
\end{aligned}$$

By the direct check we verify that the functions $g = -x_1(2\xi)^{-1}$, $h = -x_2(2\xi)^{-1}$ satisfy this system. So that we can choose

$$\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2(x_0 - x_3)^{-1}.$$

Performing the same calculations for the remaining subalgebras listed in Assertion 2.1, we arrive at the following statement.

Assertion 2.5 *Each subalgebra L_j, $(j = 1, 2, \ldots, 22)$ from the list given in Assertion 2.1 yields invariant ansatz (2.18) with*

$$\Lambda^{-1} = H = \exp(\theta_0 S_{03}) \exp(-\theta_3 S_{12}) \exp(-2\theta_1 H_1) \exp(-2\theta_2 H_2).$$

And what is more, the functions $\theta_\mu = \theta_\mu(x_0, \mathbf{x})$, $(\mu = 0, 1, 2, 3)$, $\omega = \omega(x_0, \mathbf{x})$ are given by one of the corresponding formulae below:

$$\begin{aligned}
L_1 &: \ \theta_\mu = 0,\ (\mu = 0, 1, 2, 3),\ \omega = x_3;\\
L_2 &: \ \theta_\mu = 0,\ (\mu = 0, 1, 2, 3),\ \omega = x_0;\\
L_3 &: \ \theta_\mu = 0,\ (\mu = 0, 1, 2, 3),\ \omega = \xi;\\
L_4 &: \ \theta_0 = -\ln|\xi|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha \ln|\xi|, \quad \omega = \xi \cdot \eta;\\
L_5 &: \ \theta_0 = -\ln|\xi|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = x_2;\\
L_6 &: \ \theta_0 = x_1, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = x_2;\\
L_7 &: \ \theta_0 = x_1, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = x_1 + \ln|\xi|;\\
L_8 &: \ \theta_0 = \alpha \arctan x_1 x_2^{-1}, \quad \theta_1 = \theta_2 = 0,\\
&\quad\ \theta_3 = -\arctan x_1 x_2^{-1}, \quad \omega = x_1^2 + x_2^2;\\
L_9 &: \ \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -x_0, \quad \omega = x_3;\\
L_{10} &: \ \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -(-1)^i x_3, \quad \omega = x_0;\\
L_{11} &: \ \theta_0 = \theta_1 = \theta_3 = 0, \quad \theta_2 = -\frac{(-1)^i}{2}\xi, \quad \omega = \eta;\\
L_{12} &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{2}(x_1 - \alpha x_2)\xi^{-1}, \quad \omega = \xi;\\
L_{13} &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{(-1)^i}{2}x_2, \quad \omega = \xi;\\
L_{14} &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}\xi, \quad \omega = \xi^2 - 4x_1;\\
L_{15} &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}\xi, \quad \omega = \alpha\xi^2 - 4(\alpha x_1 - x_2);\\
L_{16} &: \ \theta_0 = -\ln|\xi|,\ \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan x_1 x_2^{-1}, \quad \omega = x_1^2 + x_2^2;\\
L_{17} &: \ \theta_0 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{2}[(-1)^i x_2 + (\alpha + \xi)x_1][1 + (\alpha + \xi)\xi]^{-1},\\
&\quad\ \theta_2 = \frac{1}{2}[(-1)^i x_1 - x_2\xi][1 + (\alpha + \xi)\xi]^{-1}, \quad \omega = \xi;
\end{aligned}$$

$$
\begin{aligned}
L_{18} &: \theta_0 = -\ln|\xi|,\ \theta_1 = -\frac{1}{2}x_1\xi^{-1},\quad \theta_2 = \theta_3 = 0,\quad \omega = \xi\eta - x_1^2;\\
L_{19} &: \theta_0 = -\ln|\xi|,\ \theta_1 = -\frac{1}{2}x_1\xi^{-1},\quad \theta_2 = \theta_3 = 0,\quad \omega = x_2;\\
L_{20} &: \theta_0 = -\ln|\xi|,\ \theta_1 = -\frac{1}{2}x_1\xi^{-1},\quad \theta_2 = \theta_3 = 0,\quad \omega = \ln|\xi| + x_2;\\
L_{21} &: \theta_0 = -\ln|\xi|,\ \theta_1 = -\frac{1}{2}(x_1 + \ln|\xi|)\xi^{-1},\\
&\quad \theta_2 = \theta_3 = 0,\quad \omega = \alpha\ln|\xi| + x_2;\\
L_{22} &: \theta_0 = -\ln|\xi|,\ \theta_1 = -\frac{1}{2}x_1\xi^{-1},\quad \theta_2 = -\frac{1}{2}x_2\xi^{-1}, \theta_3 = \alpha\ln|\xi|,\\
&\quad \omega = x_\mu x^\mu\ (\mu = 0, 1, 2, 3).
\end{aligned}
$$

Here $i = 1, 2;\ \alpha \in R;\ \xi = x_0 - x_3,\ \eta = x_0 + x_3.$

2.3.2 $\tilde{P}(1,3)$-invariant ansatzes

Generically, the list of $\tilde{P}(1,3)$-invariant ansatzes is exhausted by $P(1,3)$-invariant ansatzes given in Assertion 2.5 and by ansatzes invariant with respect to the subalgebras $F_j,\ (j = 1, 2, \ldots, 24)$ listed in Assertion 2.2. By this reason, to construct all inequivalent $\tilde{P}(1,3)$-invariant ansatzes, it suffices to consider the cases of the subalgebras $F_j,\ (j = 1, 2, \ldots 24)$ only.

A preliminary analysis of these algebras shows that for the algebras F_j with j taking the values $2, 3, 4, 12, 13, \ldots, 19$ we can choose $\theta_1 = \theta_2 = \theta_4 = \theta_5 = 0$ in (2.26) and, in addition, we can put $f_1^a = f_2^a = f_4^a = f_5^a = 0$, $(a = 1, 2, 3)$ in (2.35). As a consequence, system (2.35) for the subalgebras in question reads as

$$
\xi_a^\mu \frac{\partial\theta}{\partial x_\mu} = f^a\theta,\quad \xi_a^\mu \frac{\partial\theta_0}{\partial x_\mu} = -f_0^a,\quad \xi_a^\mu \frac{\partial\theta_3}{\partial x_\mu} = f_3^a,
$$

where $\mu = 0, 1, 2, 3;\ a = 1, 2, 3.$

For the remaining subalgebras from the list given in Assertion 2.2 the following equalities hold, $\theta_b = 0, f_b^a = 0$, $(b = 2, 3, 4, 5;\ a = 1, 2, 3)$, and system (2.35) takes the form

$$
\xi_a^\mu \frac{\partial\theta}{\partial x_\mu} = f^a\theta,\quad \xi_a^\mu \frac{\partial\theta_0}{\partial x_\mu} = -f_0^a,\quad \xi_a^\mu \frac{\partial\theta_1}{\partial x_\mu} = -\theta_1 f_0^a + \frac{1}{2}f_1^a,
$$

where $\mu = 0, 1, 2, 3;\ a = 1, 2, 3.$

Summing up, we conclude that the problem of construction of $\tilde{P}(1,3)$-invariant ansatzes reduces to finding solutions of linear systems of first-order partial differential equations, that are integrated by rather standard methods of the general theory of partial differential equations.

We omit the cumbersome intermediate calculations, which are very much the same as those performed in the previous subsection, and give the final result.

Assertion 2.6 *Each subalgebra F_j, $(j = 1, 2, \dots, 24)$ from the list given in Assertion 2.2 yields invariant ansatz (2.18) with*

$$\Lambda^{-1} = H = \exp\{(-\ln\theta)E\}\exp(\theta_0 S_{03})\exp(-\theta_3 S_{12})\exp(-2\theta_1 H_1), \quad \theta_1\theta_3 = 0.$$

And what is more, the functions $\theta = \theta(x_0, \mathbf{x})$, $\theta_0 = \theta_0(x_0, \mathbf{x})$, $\theta_1 = \theta_1(x_1, \mathbf{x})$, $\theta_3 = \theta_3(x_3, \mathbf{x})$, $\omega = \omega(x_0, \mathbf{x})$ are given by one of the corresponding formulae below

$$\begin{aligned}
F_1 &: \theta = |x_1|^{-k}, \quad \theta_0 = \theta_1 = \theta_3 = 0, \quad \omega = x_2 x_1^{-1};\\
F_2 &: \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \theta_1 = 0, \quad \theta_3 = \arctan x_2 x_1^{-1},\\
&\quad \omega = \ln(x_1^2 + x_2^2) + 2\alpha \arctan x_2 x_1^{-1}, \ \alpha > 0;\\
F_3 &: \theta = |x_3|^{-k}, \theta_0 = \theta_1 = 0, \theta_3 = \arctan x_2 x_1^{-1}, \omega = (x_1^2 + x_2^2) x_3^{-2};\\
F_4 &: \theta = |x_0|^{-k}, \theta_0 = \theta_1 = 0, \theta_3 = \arctan x_2 x_1^{-1}, \omega = (x_1^2 + x_2^2) x_0^{-2};\\
F_5 &: \theta = |x_1|^{-k}, \quad \theta_0 = \alpha^{-1}\ln|x_1|, \quad \theta_1 = \theta_3 = 0, \quad \omega = x_2 x_1^{-1}, \quad \alpha > 0;\\
F_6 &: \theta = |\xi\eta|^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln|\eta\xi^{-1}|, \quad \theta_1 = \theta_3 = 0,\\
&\quad \omega = (1 - \alpha)\ln|\eta| + (1 + \alpha)\ln|\xi|, \quad \alpha > 0;\\
F_7 &: \theta = |x_2|^{-k}, \theta_0 = \alpha^{-1}\ln|x_2|, \theta_1 = \theta_3 = 0, \omega = |\xi|^{\alpha}|x_2|^{1-\alpha}, \alpha > 0;\\
F_8 &: \theta = |\eta|^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln|\eta|, \quad \theta_1 = \theta_3 = 0,\\
&\quad \omega = \xi - (-1)^j \ln|\eta|, \quad j = 1, 2;\\
F_9 &: \theta = |x_2|^{-k}, \quad \theta_0 = \ln|x_2|, \quad \theta_1 = \theta_3 = 0,\\
&\quad \omega = \xi - 2(-1)^j \ln|x_2|, \quad j = 1, 2;\\
F_{10} &: \theta = |x_2|^{-k}, \quad \theta_0 = \ln|\eta x_2^{-1}|, \quad \theta_1 = \theta_3 = 0, \quad \omega = \xi\eta x_2^{-2};\\
F_{11} &: \theta = |x_2|^{-1}, \quad \theta_0 = -\ln|\xi x_2^{-1}|, \quad \theta_1 = \theta_3 = 0, \quad \omega = x_2 x_1^{-1};\\
F_{12} &: \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = -\alpha \arctan x_2 x_1^{-1}, \quad \theta_1 = 0,\\
&\quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = \ln(x_1^2 + x_2^2) + 2\beta \arctan x_2 x_1^{-1},\\
&\quad \alpha \neq 0, \quad \beta > 0;\\
F_{13} &: \theta = |\xi\eta|^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln|\eta\xi^{-1}|, \quad \theta_1 = 0,\\
&\quad \theta_3 = -\frac{1}{2\alpha}\ln|\eta\xi^{-1}|, \quad \omega = (\alpha - \beta)\ln|\eta| + (\alpha + \beta)\ln|\xi|,\\
&\quad \alpha \neq 0, \quad \beta > 0;\\
F_{14} &: \theta = |\eta|^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln|\eta|, \quad \theta_1 = 0, \quad \theta_3 = -\frac{1}{2}\ln|\eta|, \quad \omega = \xi - \ln|\eta|;\\
F_{15} &: \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = -\alpha \arctan x_2 x_1^{-1}, \quad \theta_1 = 0,\\
&\quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = \ln(x_1^2 + x_2^2)\xi^{-2} + 2\alpha \arctan x_2 x_1^{-1}, \ \alpha \neq 0;\\
F_{16} &: \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2)\xi^{-2}, \quad \theta_1 = 0,\\
&\quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = \ln(x_1^2 + x_2^2)^{1-\alpha}\xi^{2\alpha} + 2\beta \arctan x_2 x_1^{-1},
\end{aligned}$$

$$0 \le |\alpha| \le 1, \beta \ge 0, |\alpha| + |\beta| \ne 0;$$

$$\begin{aligned}
F_{17} \quad &: \quad \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2), \quad \theta_1 = 0, \quad \theta_3 = \arctan x_2 x_1^{-1}, \\
& \quad \omega = \xi - (-1)^j \ln(x_1^2 + x_2^2) + 2\alpha \arctan x_2 x_1^{-1}, \quad \alpha \in R, \quad j = 1, 2; \\
F_{18} \quad &: \quad \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2), \quad \theta_1 = 0, \\
& \quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = \xi + 2(-1)^j \arctan x_2 x_1^{-1}, \quad j = 1, 2; \\
F_{19} \quad &: \quad \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln|\xi\eta^{-1}|, \quad \theta_1 = 0, \\
& \quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = (x_1^2 + x_2^2)(\xi\eta)^{-1}; \\
F_{20} \quad &: \quad \theta = |\xi\eta - x_1^2|^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2\alpha}\ln|\xi\eta - x_1^2|, \quad \theta_1 = -\frac{1}{2}x_1\xi^{-1}, \\
& \quad \theta_3 = 0, \quad \omega = |\xi|^{2\alpha}|\xi\eta - x_1^2|^{1-\alpha}, \ 0 \le |\alpha| \le 1; \\
F_{21} \quad &: \quad \theta = |x_1 - (-1)^j \xi x_2|^{-k}, \quad \theta_0 = \ln|x_1 - (-1)^j \xi x_2|, \quad \theta_1 = -\frac{(-1)^j}{2}x_2, \\
& \quad \theta_3 = 0, \quad \omega = \xi, \quad j = 1, 2; \\
F_{22} \quad &: \quad \theta = |\xi|^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln|\xi|, \quad \theta_1 = -\frac{1}{2}x_1\xi^{-1}, \\
& \quad \theta_3 = 0, \quad \omega = \eta - x_1^2\xi^{-1} + (-1)^j \ln|\xi|, \quad j = 1, 2; \\
F_{23} \quad &: \quad \theta = |x_2|^{-k}, \quad \theta_0 = \frac{1}{2}\ln|x_2|, \quad \theta_1 = -\frac{(-1)^j}{4}\xi^{-1}, \\
& \quad \theta_3 = 0, \quad \omega = (\xi^2 - 4(-1)^j x_1)x_2^{-1}, \quad j = 1, 2; \\
F_{24} \quad &: \quad \theta = |\xi^2 - 4(-1)^j x_1|^{-k}, \quad \theta_0 = \frac{1}{2}\ln|\xi^2 - 4(-1)^j|x_1|, \quad \theta_1 = -\frac{(-1)^j}{4}\xi, \\
& \quad \theta_3 = 0, \omega = (\eta - (-1)^j x_1\xi + \frac{1}{6}\xi^3)^2(\xi^2 - 4(-1)^j x_1)^{-3}, j = 1, 2.
\end{aligned}$$

Here k is an arbitrarily fixed constant (the conformal degree of the algebra $c(1,3)$), $\xi = x_0 - x_3,\ \eta = x_0 + x_3$.

2.3.3 $C(1,3)$-invariant ansatzes

To obtain the full description of conformally-invariant ansatzes it suffices to consider the subalgebras $C_j,\ (j = 1, 2, \ldots, 14)$ listed in Assertion 2.3.

The preliminary analysis of these subalgebras shows that we can put $\theta_4 = \theta_5 = f_4^a = f_5^a = 0$, $(a = 1, 2, 3)$ for the subalgebras $C_j,\ (j = 1, 2, \ldots, 10)$. As a result, system (2.35) corresponding to these subalgebras takes the following form:

$$\begin{aligned}
& \xi_a^\mu \frac{\partial\theta}{\partial x_\mu} = f^a\theta, \quad \xi_a^\mu \frac{\partial\theta_0}{\partial x_\mu} = -f_0^a, \quad \xi_a^\mu \frac{\partial\theta_3}{\partial x_\mu} = f_3^a, \\
& \xi_a^\mu \frac{\partial\theta_1}{\partial x_\mu} = -\theta_1 f_0^a - \theta_2 f_3^a + \frac{1}{2}f_1^a, \quad \xi_a^\mu \frac{\partial\theta_2}{\partial x_\mu} = -\theta_2 f_0^a + \theta_1 f_3^a + \frac{1}{2}f_2^a,
\end{aligned}$$

where $a = 1, 2, 3$.

Thus the problem of constructing ansatzes invariant under the subalgebras $C_j,\ (j = 1, 2, \ldots, 10)$ is again reduced to solving linear first-order partial differential equations. However, for the remaining subalgebras $C_j,\ (j = 11, 12, 13, 14)$ system (2.35) is not linear. It has been solved for the case of the spinor field in [33]. The obtained expressions for the functions are so cumbersome that they prove to be useless within the context of symmetry reduction of the conformally-invariant nonlinear Dirac equation. By this reason, we do not give here the ansatzes corresponding to the subalgebras $C_j,\ (j = 11, 12, 13, 14)$.

Assertion 2.7 *Each subalgebra $C_j,\ (j = 1, 2, \ldots, 10)$ from the list given in Assertion 2.3 yields invariant ansatz (2.18) with*

$$\Lambda^{-1} = H = \exp\{(-\ln\theta)E\}\exp(\theta_0 S_{03})\exp(-\theta_3 S_{12})\exp(-2\theta_1 H_1)$$
$$\times\exp(-2\theta_2 H_2).$$

What is more, the functions $\theta = \theta(x_0, \mathbf{x}),\ \theta_\mu = \theta_\mu(x_0, \mathbf{x}),\ (\mu = 0, 1, 2, 3),\ \omega = \omega(x_0, \mathbf{x})$ are given by one of the corresponding formulae below.

$$\begin{aligned}
C_1 \ &: \ \theta = (1+\xi^2)^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln(1+\xi^2),\\
&\quad \theta_1 = -\frac{1}{2}(x_2 + x_1\xi)(1+\xi^2)^{-1}, \quad \theta_2 = \frac{1}{2}(x_1 - \xi x_2)(1+\xi^2)^{-1},\\
&\quad \theta_3 = -\arctan\xi, \quad \omega = (x_1 - x_2\xi)(1+\xi^2)^{-1};\\
C_2 \ &: \ \theta = (1+\xi^2)^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln(1+\xi^2),\\
&\quad \theta_1 = -\frac{1}{2}(x_2 + x_1\xi)(1+\xi^2)^{-1}, \quad \theta_2 = \frac{1}{2}(x_1 - x_2\xi)(1+\xi^2)^{-1},\\
&\quad \theta_3 = -\arctan\xi, \quad \omega = (x_2 + x_1\xi)(1+\xi^2)^{-1} - \arctan\xi;\\
C_3 \ &: \ \theta = (1+\xi^2)^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln(1+\xi^2),\\
&\quad \theta_1 = -\frac{1}{2}x_1\xi(1+\xi^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\xi(1+\xi^2)^{-1},\\
&\quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = (1+\xi^2)(x_1^2 + x_2^2)^{-1};\\
C_4 \ &: \ \theta = |x_1|^{-k}, \quad \theta_0 = \ln|x_1| - \ln(1+\xi^2),\\
&\quad \theta_1 = -\frac{1}{2}x_1\xi(1+\xi^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\xi(1+\xi^2)^{-1},\\
&\quad \theta_3 = 0, \quad \omega = x_2 x_1^{-1};\\
C_5 \ &: \ \theta = ((x_1^2 + x_2^2)(1+\xi^2))^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2)(1+\xi^2)^{-1},\\
&\quad \theta_1 = -\frac{1}{2}x_1\xi(1+\xi^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\xi(1+\xi^2)^{-1},\\
&\quad \theta_3 = \arctan x_2 x_1^{-1}, \quad \omega = \arctan x_2 x_1^{-1} + \alpha\arctan\xi, \quad \alpha \neq 0;
\end{aligned}$$

$$
\begin{aligned}
C_6 \ \ : \ \ & \theta = [(x_1 - x_2\xi)^2(1+\xi^2)^{-1}]^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln[(x_1 - x_2\xi)^2(1+\xi^2)^{-3}], \\
& \theta_1 = -\frac{1}{2}(x_2 + x_1\xi)(1+\xi^2)^{-1}, \quad \theta_2 = \frac{1}{2}(x_1 - x_2\xi)(1+\xi^2)^{-1}, \\
& \theta_3 = -\arctan\xi, \quad \omega = \alpha\arctan\xi - \ln[(x_1 - x_2\xi)(1+\xi^2)^{-1}], \quad \alpha \neq 0; \\
C_7 \ \ : \ \ & \theta = [(x_1 - x_2\xi)^2(1+\xi^2)^{-1}]^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln[(x_1 - x_2\xi)^2(1+\xi^2)^{-3}], \\
& \theta_1 = -\frac{1}{2}(x_2 + x_1\xi)(1+\xi^2)^{-1}, \quad \theta_2 = \frac{1}{2}(x_1 - x_2\xi)(1+\xi^2)^{-1}, \\
& \theta_3 = -\arctan\xi, \\
& \omega = [\eta(1+\xi^2)^2 - 2x_1(x_2 + x_1\xi) - \xi(x_1^2\xi^2 - x_2^2)][x_1 - \xi x_2]^{-2} - \xi; \\
C_8 \ \ : \ \ & \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln[(x_1^2 + x_2^2)(1+\xi^2)^{-2}], \\
& \theta_1 = -\frac{1}{2}x_1\xi(1+\xi^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\xi(1+\xi^2)^{-1}, \\
& \theta_3 = \arctan x_2x_1^{-1}, \\
& \omega = \ln(x_1^2 + x_2^2)(1+\xi^2)^{-1} + 2\alpha\arctan x_2x_1^{-1} - 2\beta\arctan\xi, \\
& \alpha, \beta \in R, \quad |\alpha| + |\beta| \neq 0; \\
C_9 \ \ : \ \ & \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = \frac{1}{2}\ln(x_1^2 + x_2^2) - \ln(1+\xi^2), \\
& \theta_1 = -\frac{1}{2}x_1\xi(1+\xi^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\xi(1+\xi^2)^{-1}, \\
& \theta_3 = \arctan x_2x_1^{-1}, \quad \omega = \eta(1+\xi^2)(x_1^2 + x_2^2)^{-1} - \xi; \\
C_{10} \ \ : \ \ & \theta = (x_1^2 + x_2^2)^{-\frac{k}{2}}, \quad \theta_0 = -\frac{1}{2}\ln(x_1^2 + x_2^2), \\
& \theta_1 = -\frac{1}{2}x_1\eta(x_1^2 + x_2^2)^{-1}, \quad \theta_2 = -\frac{1}{2}x_2\eta(x_1^2 + x_2^2)^{-1}, \\
& \theta_3 = 0, \quad \omega = x_2x_1^{-1}.
\end{aligned}
$$

Here k is an arbitrarily fixed constant (the conformal degree of the algebra $c(1,3)$), $\xi = x_0 - x_3, \eta = x_0 + x_3$.

3 Exact solutions of the Yang-Mills equations

In this section we apply the above described technique in order to perform in-depth analysis of the problems of symmetry reduction and construction of exact invariant solutions of the $SU(2)$ Yang-Mills equations in the (1+3) dimensional Minkowski space of independent variables. Since the general method to be used relies heavily upon symmetry properties of the equations under study, we will review briefly the group-theoretical properties of the $SU(2)$ Yang-Mills equations.

3.1 Symmetry properties of the Yang-Mills equations

The classical Yang-Mills equations of $SU(2)$ gauge theory in the Minkowski space-time $R^{1,3}$ form the system of twelve nonlinear second-order partial differential equations of the form

$$\begin{aligned}&\partial_\nu\partial^\nu \mathbf{A}_\mu - \partial^\mu\partial_\nu \mathbf{A}_\nu + e[(\partial_\nu \mathbf{A}_\nu)\times \mathbf{A}_\mu - 2(\partial_\nu \mathbf{A}_\mu)\times \mathbf{A}_\nu \\ &\quad +(\partial^\mu \mathbf{A}_\nu)\times \mathbf{A}^\nu] + e^2 \mathbf{A}_\nu \times (\mathbf{A}^\nu \times \mathbf{A}_\mu) = 0.\end{aligned} \tag{3.1}$$

Hereafter in this section, the indices $\mu,\nu,\alpha,\beta,\gamma,\delta,\sigma$ take the values $0,1,2,3$; $\partial_\mu = \partial_{x_\mu} = \frac{\partial}{\partial x_\mu}$; rising and lowering the indices is performed with the use of the metric tensor $g_{\mu\nu}$ of the Minkowski space and the summation convention over the repeated indices is used. Furthermore, $\mathbf{A}_\mu = \mathbf{A}_\mu(x_0,\mathbf{x}) = (A^1_\mu(x_0,\mathbf{x}), A^2_\mu(x_0,\mathbf{x}), A^3_\mu(x_0,\mathbf{x}))^{\mathbf{T}}$ is the vector-potential of the Yang-Mills field (for brevity it is called in the sequel the Yang-Mills field) and e is the gauge coupling constant.

The maximal symmetry group admitted by equations (3.1) is the group $C(1,3)\otimes SU(2)$ [17], where $C(1,3)$ is the 15-parameter conformal group generated by the following vector fields:

$$\begin{aligned}P_\mu &= \partial_{x_\mu}, \\ J_{\mu\nu} &= x^\mu\partial_{x_\nu} - x^\nu\partial_{x_\mu} + A^{a\mu}\partial_{A^a_\nu} - \partial A^{a\nu}\partial_{A^a_\mu}, \\ D &= x_\mu\partial_{x_\mu} - A^a_\mu\partial_{A^a_\mu}, \\ K_\mu &= 2x^\mu D - (x_\nu x^\nu)\partial_{x_\mu} + 2A^{a\mu}x_\nu\partial_{A^a_\nu} - 2A^a_\nu x^\nu\partial_{A^a_\mu}\end{aligned} \tag{3.2}$$

and $SU(2)$ is the infinite-parameter unitary gauge transformation group having the generator

$$Q = \left(\varepsilon_{abc}A^b_\mu\omega^c(x_0,\mathbf{x}) + e^{-1}\partial_{x_\mu}\omega^a(x_0,\mathbf{x})\right)\partial_{A^a_\mu}. \tag{3.3}$$

In formulae (3.2), (3.3), $\partial_{A^a_\mu} = \frac{\partial}{\partial A^a_\mu}$, $\omega^c(x_0,\mathbf{x})$ stand for arbitrary real functions, $a,b,c = 1,2,3$ and ε_{abc} is the anti-symmetric third-order tensor with $\varepsilon_{123}=1$.

It is not difficult to check that vector fields (3.2) can be rewritten in the form (2.11) if we put

$$S_{01} = \begin{pmatrix} 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ S_{02} = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{03} = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},\ S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

$$S_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where 0 and I are the zero and unit 3×3 matrices, correspondingly. Next, we choose the matrix E to be the 12×12 unit matrix and the conformal degree k of the algebra $c(1,3)$ to be equal to 1.

One of the important applications of the symmetry admitted by the Yang-Mills equations is a possibility of getting new exact solutions with the help of the solution generation formulae. This method is based upon the fact that the symmetry group maps the set of solutions of an equation admitting this group into itself. We give the corresponding formulae without proof (see, [20, 21, 33] for further details).

Assertion 3.1 *Let*

$$\begin{aligned} \overline{x}_i &= f_i(\mathbf{x}, \mathbf{u}, \tau), \quad i = 1, 2, \dots, p, \\ \overline{u}_j &= g_j(\mathbf{x}, \mathbf{u}, \tau), \quad j = 1, 2, \dots, q, \end{aligned}$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_r)$, *be the* r*-parameter invariance group admitted by a system of partial differential equations and* $U_j(\mathbf{x})$, $j = 1, 2, \dots, q$ *be a particular solution of the latter. Then the* q*-component function* $\mathbf{u}(\mathbf{x}) = (u^1(x), \dots, u^q(x))$, *defined in implicit way by the formulae*

$$U_j(\mathbf{f}(\mathbf{x}, \mathbf{u}, \tau)) = g_j(\mathbf{x}, \mathbf{u}, \tau)$$

with $\mathbf{f} = (f_1, \dots, f_p)$, $j = 1, 2, \dots, q$, *is also a solution of the system in question.*

In order to be able to take advantage of Assertion 3.1, we need the formulae for the final transformations generated by the basis operators (3.2), (3.3) of the Lie algebra of the group $C(1,3) \otimes SU(2)$. We give these formulae following [2, 21].

1) the translation group (the generator is $X = \tau_\mu P_\mu$)

$$\overline{x}_\mu = x_\mu + \tau_\mu, \overline{A}^d_\mu = A^d_\mu;$$

2) the Lorentz group $O(1,3)$

   (a) the rotation group (the generator is $X = \tau J_{ab}$)

$$\begin{aligned} \overline{x}_0 &= x_0, \quad \overline{x}_c = x_c, \quad c \neq a, \quad c \neq b, \\ \overline{x}_a &= x_a \cos\tau + x_b \sin\tau, \\ \overline{x}_b &= x_b \cos\tau - x_a \sin\tau, \\ \overline{A}^d_0 &= A^d_0, \overline{A}^d_c = A^d_c, \quad c \neq a, \ c \neq b, \\ \overline{A}^d_a &= A^d_a \cos\tau + A^d_b \sin\tau, \\ \overline{A}^d_b &= A^d_b \cos\tau - A^d_a \sin\tau; \end{aligned}$$

(b) the Lorentz transformations (the generator is $X = \tau J_{0a}$)

$$\begin{aligned}
\overline{x}_0 &= x_0 \cosh\tau + x_a \sinh\tau, \\
\overline{x}_a &= x_a \cosh\tau + x_0 \sinh\tau, \\
\overline{A}_0^d &= A_0^d \cosh\tau + A_a^d \sinh\tau, \\
\overline{A}_a^d &= A_a^d \cosh\tau + A_0^d \sinh\tau, \\
\overline{x}_b &= x_b, \quad \overline{A}_b^d = A_b^d, \quad b \neq a;
\end{aligned}$$

3) the scale transformation group (the generator is $X = \tau D$)

$$\overline{x}_\mu = x_\mu\, e^\tau, \quad \overline{A}_\mu^d = A_\mu^d e^{-\tau}.$$

4) the group of conformal transformations (the generation is $X = \tau_\mu K^\mu$)

$$\begin{aligned}
\overline{x}_\mu &= (x_\mu - \tau_\mu x_\nu x^\nu)\sigma^{-1}(x_0, \mathbf{x}), \\
\overline{A}_\mu^d &= [g_{\mu\nu}\sigma(x_0, \mathbf{x}) + 2(x_\mu \tau_\nu - x_\nu \tau_\mu \\
&\quad +2\tau_\alpha x^\alpha \tau_\mu x_\nu - x_\alpha x^\alpha \tau_\mu \tau_\nu - \tau_\alpha \tau^\alpha x_\mu x_\nu] A^{d\nu}.
\end{aligned}$$

5) the gauge transformation group (the generator is $X = Q$)

$$\begin{aligned}
\overline{x}_\mu &= x_\mu, \\
\overline{A}_\mu^d &= A_\mu^d \cos\omega + \varepsilon_{dbc} A_\mu^b n^c \sin\omega + 2n^d n^b A_\mu^b \sin^2\frac{\omega}{2} \\
&\quad + e^{-1}[\frac{1}{2} n^d \partial_{x_\mu}\omega + \frac{1}{2}(\partial_{x_\mu} n^d)\sin\omega + \varepsilon_{dbc}(\partial_{x_\mu} n^b) n^c].
\end{aligned}$$

In the above formulae $\sigma(x_0, \mathbf{x}) = 1 - \tau_\alpha \tau^\alpha + (\tau_\alpha \tau^\alpha)(x_\beta x^\beta)$, $n^a = n^a(x_0, \mathbf{x})$ are the components of the unit vector given by the relations $\omega^a(x_0, \mathbf{x}) = \omega(x_0, \mathbf{x}) n^a(x_0, \mathbf{x})$ with $a, b, c, d = 1, 2, 3$.

Using Assertion 3.1, it is not difficult to derive the formulae for generating solutions of the Yang-Mills equations by the above enumerated transformation groups. We give these following [33].

1) the translation group

$$A_\mu^a(x) = u_\mu^a(x + \tau);$$

2) the Lorentz group

$$\begin{aligned}
A_\mu^d(x) &= a_\mu u_0^d(ax, bx, cx, dx) + b_\mu u_1^d(ax, bx, cx, dx) \\
&\quad + c_\mu u_2^d(ax, bx, cx, dx) + d_\mu u_3^d(ax, bx, cx, dx);
\end{aligned}$$

3) the scale transformation group

$$A^d_\mu(x) = e^\tau u^d_\mu(xe^\tau);$$

4) the group of conformal transformations

$$\begin{aligned} A^d_\mu(x) &= [g_{\mu\nu}\sigma^{-1}(x) + 2\sigma^{-2}(x)(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha \tau_\mu x_\nu \\ &\quad -x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu)]u^{d\nu}((x - \tau(x_\alpha x^\alpha))\sigma^{-1}(x)). \end{aligned}$$

5) the gauge transformation group

$$\begin{aligned} A^d_\mu(x) &= u^d_\mu \cos\omega + \varepsilon_{dbc} u^b_\mu n^c \sin\omega + 2n^d n^b u^b_\mu \sin^2\frac{\omega}{2} \\ &\quad +e^{-1}[\frac{1}{2}n^d \partial_{x_\mu}\omega + \frac{1}{2}(\partial_{x_\mu} n^d)\sin\omega + \varepsilon_{dbc}(\partial_{x_\mu} n^b)n^c]. \end{aligned}$$

Here $u^d_\mu(x)$is an arbitrary particular solution of the Yang-Mills equations; $x = (x_0, \mathbf{x})$; τ, τ_μ are arbitrary parameters; $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary constants satisfying the relations

$$\begin{aligned} a_\mu a^\mu &= -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ a_\mu b^\mu &= a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned} \tag{3.5}$$

In addition, we use the following notations:

$$\begin{aligned} & x + \tau = \{x_\mu + \tau_\mu, \mu = 0, 1, 2, 3\}, \\ & ax = a_\mu x^\mu, \quad bx = b_\mu x^\mu, \quad cx = c_\mu x^\mu, \quad dx = d_\mu x^\mu. \end{aligned}$$

Thus, using the solution generation formulae enables extending a single solution of the Yang-Mills equations to a multi-parameter family of exact solutions.

Let us also discuss briefly the discrete symmetries of equations (3.1). It is straightforward to check that the Yang-Mills equations admit the following groups of discrete transformations:

$$\begin{aligned} \Psi_1 &: \ \overline{x}_\mu = -x_\mu, \ \ \overline{\mathbf{A}}_\mu = -\mathbf{A}_\mu; \\ \Psi_2 &: \ \overline{x}_0 = -x_0, \ \ \overline{x}_1 = -x_1, \overline{x}_2 = x_2, \ \overline{x}_3 = x_3, \\ &\quad \ \overline{\mathbf{A}}_0 = \mathbf{A}_0, \ \overline{\mathbf{A}}_1 = -\mathbf{A}_1, \ \overline{\mathbf{A}}_2 = \mathbf{A}_2, \ \overline{\mathbf{A}}_3 = \mathbf{A}_3; \\ \Psi_3 &: \ \overline{x}_0 = -x_0, \ \ \overline{x}_1 = x_1, \overline{x}_2 = -x_2, \ \overline{x}_3 = -x_3, \\ &\quad \ \overline{\mathbf{A}}_0 = -\mathbf{A}_0, \ \overline{\mathbf{A}}_1 = \mathbf{A}_1, \ \overline{\mathbf{A}}_2 = \mathbf{A}_2, \ \overline{\mathbf{A}}_3 = -\mathbf{A}_3. \end{aligned}$$

Action of these transformation groups on the basis elements (3.2) of the symmetry algebra admitted by equations (3.1) is described in Table 3.1, where $G_m = J_{0m} - J_{m3}$, $(m = 1, 2)$, $M = P_0 + P_3$, $T = \frac{1}{2}(P_0 - P_3)$.

While classifying the subalgebras of the algebras $p(1,3)$ and $\tilde{p}(1,3)$ of the rank 3 we have exploited the discrete symmetries Φ_a given in Table 2.1. Comparing Tables 2.1 and 3.1 we see that the actions of the discrete symmetries Φ_a and Ψ_a on the operators $P_\mu, J_{\mu\nu}, D$ give identical results, namely,

$$\Phi_a P_\mu = \Psi_a P_\mu, \quad \Phi_a J_{\mu\nu} = \Psi_a J_{\mu\nu}, \quad \Phi_a D = \Psi_a D$$

for all $a = 1,2,3$. This fact makes it possible to use the discrete symmetries in order to simplify the forms of the basis operators of subalgebras of the algebras $p(1,3)$, $\tilde{p}(1,3)$.

Table 3.1. Discrete symmetries of equations (3.1)

Generators	Action of Ψ_a		
	Ψ_1	Ψ_2	Ψ_3
P_0	$-P_0$	P_0	$-P_0$
P_1	$-P_1$	$-P_1$	P_1
P_k $(k=2,3)$	$-P_k$	P_k	$-P_k$
J_{03}	J_{03}	J_{03}	J_{03}
J_{12}	J_{12}	$-J_{12}$	$-J_{12}$
G_1	G_1	$-G_1$	$-G_1$
G_2	G_2	G_2	G_2
M	$-M$	M	$-M$
T	$-T$	T	$-T$
D	D	D	D
K_0	$-K_0$	K_0	$-K_0$
K_1	$-K_1$	$-K_1$	K_1
K_m $(m=2,3)$	$-K_m$	K_m	$-K_m$

3.2 Ansatzes for the Yang-Mills field

Conformally-invariant ansatzes for the Yang-Mills field, that reduce equations (3.1) to systems of ordinary differential equations, can be represented in the linear form

$$\mathbf{A}_\mu(x_0, \mathbf{x}) = \Lambda_{\mu\nu}\mathbf{B}_\nu(\omega), \tag{3.6}$$

where $\Lambda_{\mu\nu} = \Lambda_{\mu\nu}(x_0, \mathbf{x})$ are some fixed non-singular 3×3 matrices and $\mathbf{B}_\nu(\omega) = (B^1_\nu(\omega),\ B^2_\nu(\omega), B^3_\nu(\omega))^T$ are new unknown vector functions of the new independent variable $\omega = \omega(x_0, \mathbf{x})$. In the sequel, we will denote the 12×12 matrix having the matrix entries $\Lambda_{\mu\nu}$ as Λ.

Due to the space limitations, we restrict our considerations to the ansatzes invariant under the subalgebras of the Poincaré algebra. Let us note that the case of the extended Poincaré algebra is handled in [39].

The structure of the matrix Λ for the case of arbitrary vector field is described in Assertion 2.5. Adapting the formula for Λ to the case in hand, we have

$$\Lambda = \exp(2\theta_1 H_1)\exp(2\theta_2 H_2)\exp(-\theta_0 S_{03})\exp(\theta_3 S_{12}),$$

where $\theta_\mu = \theta_\mu(x_0, \mathbf{x})$ are some real-valued functions, $H_1 = S_{01} - S_{13}, H_2 = S_{02} - S_{23}$ and $S_{\mu\nu}$ are matrices (3.4), that realize the matrix representation of the Lie algebra $o(1,3)$ of the Lorentz group $O(1,3)$.

Computing the exponents with the help of the Campbell-Hausdorff formula yields

$$\Lambda = \begin{pmatrix} [\cosh\theta_0 + \Phi] & -2[\Psi_1] & 2[\Psi_2] & [\sinh\theta_0 - \Phi] \\ [-2\theta_1 e^{-\theta_0}] & [\cos\theta_3] & [-\sin\theta_3] & [2\theta_1 e^{-\theta_0}] \\ [-2\theta_2 e^{-\theta_0}] & [\sin\theta_3] & [\cos\theta_3] & [2\theta_2 e^{-\theta_0}] \\ [\sinh\theta_0 + \Phi] & -2[\Psi_1] & 2[\Psi_2] & [\cosh\theta_0 + \Phi] \end{pmatrix},$$

where $\Phi = 2(\theta_1^2 + \theta_2^2)e^{-\theta_0}$, $\Psi_1 = \theta_1\cos\theta_3 + \theta_2\sin\theta_3$, $\Psi_2 = \theta_1\sin\theta_3 - \theta_2\cos\theta_3$ and the symbol $[f]$ stands for fI, I being the unit 3×3 matrix.

Inserting the obtained expression for the matrix Λ into (3.6) yields the final form of the Poincaré-invariant ansatz for the Yang-Mills field

$$\begin{aligned}
\mathbf{A}_0 &= \cosh\theta_0\mathbf{B}_0 + \sinh\theta_0\mathbf{B}_3 - 2(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)\mathbf{B}_1 \\
&\quad +2(\theta_1\sin\theta_3 - \theta_2\cos\theta_3)\mathbf{B}_2 + 2(\theta_1^2+\theta_2^2)e^{-\theta_0}(\mathbf{B}_0 - \mathbf{B}_3), \\
\mathbf{A}_1 &= \cos\theta_3\mathbf{B}_1 - \sin\theta_3\mathbf{B}_2 - 2\theta_1 e^{-\theta_0}(\mathbf{B}_0 - \mathbf{B}_3), \\
\mathbf{A}_2 &= \sin\theta_3\mathbf{B}_1 + \cos\theta_3\mathbf{B}_2 - 2\theta_2 e^{-\theta_0}(\mathbf{B}_0 - \mathbf{B}_3), \\
\mathbf{A}_3 &= \sinh\theta_0\mathbf{B}_0 + \cosh\theta_0\mathbf{B}_3 - 2(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)\mathbf{B}_1 \\
&\quad +2(\theta_1\sin\theta_3 - \theta_2\cos\theta_3)\mathbf{B}_2 + 2(\theta_1^2+\theta_2^2)e^{-\theta_0}(\mathbf{B}_0 - \mathbf{B}_3),
\end{aligned} \tag{3.7}$$

where $\mathbf{B}_\mu = \mathbf{B}_\mu(\omega)$ and the forms of the functions θ_μ, ω are given in Assertion 2.5.

Inserting (3.7) into (3.1) yields a system of ordinary differential equations for the functions $\mathbf{B}_\mu(\omega)$. If we will succeed in constructing its general or particular solution, then substituting it into (3.7) gives an exact solution of the Yang-Mills equations (3.1). However, the so constructed solution will have an unpleasant feature of being asymmetric in the variables x_μ, while equations (3.1) are symmetric in these.

To get exact solutions, that are symmetric in all the variables, we exploit the formulae for generating solutions by Lorentz transformations (see, the previous subsection) and thus come to the following general form of the Pojncaré-invariant ansatz:

$$\mathbf{A}_\mu(x) = a_{\mu\nu}(x)\mathbf{B}^\nu(\omega), \tag{3.8}$$

where

$$
\begin{aligned}
a_{\mu\nu}(x) \quad = \quad & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 \\
& +2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu \\
& +(\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 \qquad (3.9) \\
& -(c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu).
\end{aligned}
$$

Here $\mu,\nu = 0,1,2,3$; $x = (x_0,\mathbf{x})$ and $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary parameters that satisfy relations (3.5). Thus, we have represented Poincaré-invariant ansatzes (3.7) in the explicitly covariant form.

Before giving the corresponding forms of the functions θ_μ, ω for the above ansatz, we remind that using the discrete symmetries Φ_a, $(a = 1,2,3)$ enables simplifying the forms of the subalgebras of the algebra $p(1,3)$. Namely, at the expense of these symmetries we can put $j = 2$ in the subalgebras L_i^j, $(i = 10, 11, 13, 17)$. Consequently, for the corresponding ansatzes we have $(-1)^j = 1$. With this remark the forms of the functions θ_μ, ω, defining ansatzes (3.8), (3.9) invariant with respect to subalgebras from Assertion 2.1, read as

$$
\begin{aligned}
L_1 \ &: \ \theta_\mu = 0, \quad \omega = dx; \\
L_2 \ &: \ \theta_\mu = 0, \quad \omega = ax; \\
L_3 \ &: \ \theta_\mu = 0, \quad \omega = kx; \\
L_4 \ &: \ \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha\ln|kx|, \quad \omega = (ax)^2 - (dx)^2; \\
L_5 \ &: \ \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = cx; \\
L_6 \ &: \ \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \ \omega = cx; \\
L_7 \ &: \ \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = bx - \ln|kx|; \\
L_8 \ &: \ \theta_0 = \alpha\arctan(bx(cx)^{-1}), \quad \theta_1 = \theta_2 = 0, \\
&\quad\ \theta_3 = -\arctan(bx(cx)^{-1}), \quad \omega = (bx)^2 + (cx)^2; \\
L_9 \ &: \ \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad \omega = dx; \\
L_{10} \ &: \ \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad \omega = ax; \\
L_{11} \ &: \ \theta_0 = \theta_1 = \theta_3 = 0, \quad \theta_2 = -\frac{1}{2}kx, \quad \omega = ax - dx; \\
L_{12} \ &: \ \theta_0 = 0, \quad \theta_1 = \frac{1}{2}(bx - \alpha cx)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad \omega = kx; \\
L_{13} \ &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}cx, \quad \omega = kx; \\
L_{14} \ &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}kx, \quad \omega = 4bx + (kx)^2; \\
L_{15} \ &: \ \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}kx, \quad \omega = 4(\alpha bx - cx) + \alpha(kx)^2; \\
L_{16} \ &: \ \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(bx(cx)^{-1}), \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
& \omega=(bx)^2+(cx)^2; \\
L_{17} \quad : \quad & \theta_0=\theta_3=0, \quad \theta_1=\frac{1}{2}(cx+(\alpha+kx)bx)(1+kx(\alpha+kx))^{-1}, \\
& \theta_2=-\frac{1}{2}(bx-cx\cdot kx)(1+kx(\alpha+kx))^{-1}, \quad \omega=kx; \\
L_{18} \quad : \quad & \theta_0=-\ln|kx|, \quad \theta_1=\frac{1}{2}bx(kx)^{-1}, \\
& \theta_2=\theta_3=0, \quad \omega=(ax)^2-(bx)^2-(dx)^2; \\
L_{19} \quad : \quad & \theta_0=-\ln|kx|, \quad \theta_1=\frac{1}{2}bx(kx)^{-1}, \quad \theta_2=\theta_3=0, \quad \omega=cx; \\
L_{20} \quad : \quad & \theta_0=-\ln|kx|, \quad \theta_1=\frac{1}{2}bx(kx)^{-1}, \\
& \theta_2=\theta_3=0, \quad \omega=\ln|kx|-cx; \\
L_{21} \quad : \quad & \theta_0=-\ln|kx|, \quad \theta_1=\frac{1}{2}(bx-\ln|kx|)(kx)^{-1}, \\
& \theta_2=\theta_3=0, \quad \omega=\alpha\ln|kx|-cx; \\
L_{22} \quad : \quad & \theta_0=-\ln|kx|, \quad \theta_1=-\frac{1}{2}bx(kx)^{-1}, \quad \theta_2=-\frac{1}{2}cx(kx)^{-1}, \\
& \theta_3=\alpha\ln|kx|, \quad \omega=(ax)^2-(bx)^2-(cx)^2-(dx)^2.
\end{aligned}
$$

As earlier, we use the short-hand notations for the scalar product in the Minkowski space:

$$
ax=a_\mu x^\mu,\ bx=b_\mu x^\mu,\ cx=c_\mu x^\mu,\ dx=d_\mu x^\mu,
$$

and what is more, $kx=ax+dx$.

3.3 Symmetry reduction of the Yang-Mills equations

Ansatzes (3.8)–(3.10) are given in the explicitly covariant form. This fact enables us to perform symmetry reduction of equations (3.1) in a unified way. First of all, we give without derivation three important identities for the tensor $a_{\mu\nu}$ (see, e.g., [35])

$$
a_\mu^\gamma a_{\gamma\nu}=g_{\mu\nu}, \tag{3.11}
$$

$$
\begin{aligned}
a_\mu^\gamma \frac{\partial a_{\gamma\nu}}{\partial x_\delta} = & -(a_\mu d_\nu - a_\nu d_\mu)\frac{\partial\theta_0}{\partial x_\delta}+(b_\mu c_\nu - c_\mu b_\nu)\frac{\partial\theta_3}{\partial x_\delta} \\
& +2e^{-\theta_0}[(k_\mu b_\nu - k_\nu b_\mu)\cos\theta_3-(k_\mu c_\nu - k_\nu c_\mu)\sin\theta_3]\frac{\partial\theta_1}{\partial x_\delta} \\
& +2e^{-\theta_0}[(k_\mu b_\nu - k_\nu b_\mu)\sin\theta_3+(k_\mu c_\nu - k_\nu c_\mu)\cos\theta_3]\frac{\partial\theta_2}{\partial x_\delta},
\end{aligned} \tag{3.12}
$$

$$
a_\mu^\gamma \Box a_{\gamma\nu}=(a_\mu a_\nu - d_\mu d_\nu)\frac{\partial\theta_0}{\partial x_\gamma}\frac{\partial\theta_0}{\partial x^\gamma}-(a_\mu d_\nu - a_\nu d_\mu)\Box\theta_0
$$

$$
\begin{aligned}
&+2e^{-\theta_0}k_\mu b_\nu[(\Box\theta_1)\cos\theta_3+(\Box\theta_2)\sin\theta_3-2\frac{\partial\theta_1}{\partial x_\gamma}\frac{\partial\theta_3}{\partial x^\gamma}\sin\theta_3\\
&+2\frac{\partial\theta_2}{\partial x_\gamma}\frac{\partial\theta_3}{\partial x^\gamma}\cos\theta_3]+2e^{-\theta_0}k_\mu c_\nu[(\Box\theta_2)\cos\theta_3-(\Box\theta_1)\sin\theta_3\\
&-2\frac{\partial\theta_1}{\partial x_\gamma}\frac{\partial\theta_3}{\partial x^\gamma}\cos\theta_3-2\frac{\partial\theta_2}{\partial x_\gamma}\frac{\partial\theta_3}{\partial x^\gamma}\sin\theta_3]+4e^{-2\theta_0}k_\mu k_\nu \qquad (3.13)\\
&\times(\frac{\partial\theta_1}{\partial x_\gamma}\frac{\partial\theta_1}{\partial x^\gamma}+\frac{\partial\theta_2}{\partial x_\gamma}\frac{\partial\theta_2}{\partial x^\gamma})+(b_\mu b_\nu+c_\mu c_\nu)\frac{\partial\theta_3}{\partial x_\gamma}\frac{\partial\theta_3}{\partial x^\gamma}\\
&+(b_\mu c_\nu-c_\mu b_\nu)\Box\theta_3.
\end{aligned}
$$

Hereafter, we denote the derivatives of the functions in one variable ω by the dots over the symbols of the functions, for example,

$$
\frac{df}{d\omega}=\dot f,\quad \frac{d^2f}{d\omega^2}=\ddot f.
$$

.

Assertion 3.2 *Let ansatz (3.8) reduce system (3.1) to a system of second-order ordinary differential equations. Then the reduced system is necessarily of the form*

$$
\begin{aligned}
&k_{\mu\gamma}\ddot{\mathbf{B}}^\gamma+l_{\mu\gamma}\dot{\mathbf{B}}^\gamma+m_{\mu\gamma}\mathbf{B}^\gamma+eg_{\mu\nu\gamma}\dot{\mathbf{B}}^\nu\times\mathbf{B}^\gamma \qquad (3.14)\\
&+eh_{\mu\nu\gamma}\mathbf{B}^\nu\times\mathbf{B}^\gamma+e^2\mathbf{B}_\gamma\times(\mathbf{B}^\gamma\times\mathbf{B}_\mu)=0,
\end{aligned}
$$

its coefficients being given by the relations

$$
\begin{aligned}
&k_{\mu\gamma}=g_{\mu\gamma}F_1-G_\mu G_\gamma,\quad l_{\mu\gamma}=g_{\mu\gamma}F_2+2S_{\mu\gamma}-G_\mu H_\gamma-G_\mu\dot G_\gamma,\\
&m_{\mu\gamma}=R_{\mu\gamma}-G_\mu\dot H_\gamma,\quad g_{\mu\nu\gamma}=g_{\mu\gamma}G_\nu+g_{\nu\gamma}G_\mu-2g_{\mu\nu}G_\gamma, \qquad (3.15)\\
&h_{\mu\nu\gamma}=\frac{1}{2}(g_{\mu\gamma}H_\nu-g_{\mu\nu}H_\gamma)-T_{\mu\nu\gamma},
\end{aligned}
$$

where $F_1,F_2,G_\mu,H_\mu,S_{\mu\nu},R_{\mu\nu},T_{\mu\nu\gamma}$ are functions of ω presented below,

$$
\begin{aligned}
&F_1=\frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu},\quad F_2=\Box\omega,\quad G_\mu=a_{\gamma\mu}\frac{\partial\omega}{\partial x_\gamma},\\
&H_\mu=\frac{\partial a_{\gamma\mu}}{\partial x_\gamma},\quad S_{\mu\nu}=a^\gamma_\mu\frac{\partial a_{\gamma\nu}}{\partial x_\delta}\frac{\partial\omega}{\partial x^\delta},\quad R_{\mu\nu}=a^\gamma_\mu\Box a_{\gamma\nu}, \qquad (3.16)\\
&T_{\mu\nu\gamma}=a^\delta_\mu\frac{\partial a_{\delta\nu}}{\partial x_\sigma}a_{\sigma\gamma}+a^\delta_\nu\frac{\partial a_{\delta\gamma}}{\partial x_\sigma}a_{\sigma\mu}+a^\delta_\gamma\frac{\partial a_{\delta\mu}}{\partial x_\sigma}a_{\sigma\nu}.
\end{aligned}
$$

Proof. Inserting ansatz (3.8) into equation (3.1) and performing some simplifications yield the following identities:

$$
\Box\mathbf{A}_\mu-\partial^\mu(\partial_\nu\mathbf{A}_\nu)=\Big(\Box a_{\mu\gamma}-\frac{\partial^2 a_{\nu\gamma}}{\partial x^\mu\partial x_\nu}\Big)\mathbf{B}^\gamma
$$

$$+(2\frac{\partial a_{\mu\gamma}}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} + a_{\mu\gamma}\Box\omega - \frac{\partial a_{\nu\gamma}}{\partial x_\nu}\frac{\partial \omega}{\partial x^\mu} - \frac{\partial a_{\nu\gamma}}{\partial x^\mu}\frac{\partial \omega}{\partial x_\nu} \tag{3.17}$$

$$-a_{\nu\gamma}\frac{\partial^2\omega}{\partial x_\nu \partial x^\mu})\dot{\mathbf{B}}^\gamma + \left(a_{\mu\gamma}\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - a_{\nu\gamma}\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\mu}\right)\ddot{\mathbf{B}}^\gamma;$$

$$(\partial_\nu \mathbf{A}_\nu) \times \mathbf{A}_\mu - 2(\partial_\nu \mathbf{A}_\mu) \times \mathbf{A}_\nu + (\partial^\mu \mathbf{A}_\nu) \times \mathbf{A}^\nu$$

$$= (a_{\mu\gamma}\frac{\partial a_{\nu\alpha}}{\partial x_\nu} - 2\frac{\partial a_{\mu\alpha}}{\partial x_\nu}a_{\nu\gamma} + \frac{\partial a_{\nu\alpha}}{\partial x^\mu}a^\nu_\gamma)\mathbf{B}^\alpha \times \mathbf{B}^\gamma \tag{3.18}$$

$$+(a_{\mu\gamma}a_{\nu\alpha}\frac{\partial \omega}{\partial x_\nu} - 2a_{\mu\alpha}a_{\nu\gamma}\frac{\partial \omega}{\partial x_\nu} + a_{\nu\alpha}a^\nu_\gamma\frac{\partial \omega}{\partial x^\mu})\dot{\mathbf{B}}^\alpha \times \mathbf{B}^\gamma;$$

$$\mathbf{A}_\nu \times (\mathbf{A}^\nu \times \mathbf{A}_\mu) = a_{\nu\beta}a^\nu_\alpha a_{\mu\gamma}\mathbf{B}^\beta \times (\mathbf{B}^\alpha \times \mathbf{B}^\gamma). \tag{3.19}$$

Here $\alpha, \beta = 0, 1, 2, 3$.

Convoluting the left- and right-hand sides of the obtained expressions with a^μ_δ and taking into account (3.11) yield

$$a^\mu_\delta \mathbf{A}_\nu \times (\mathbf{A}^\nu \times \mathbf{A}_\mu) = a^\mu_\delta a_{\nu\beta}a^\nu_\alpha a_{\mu\gamma}\mathbf{B}^\beta \times (\mathbf{B}^\alpha \times \mathbf{B}^\gamma)$$
$$= g_{\beta\alpha}g_{\delta\gamma}\mathbf{B}^\beta \times (\mathbf{B}^\alpha \times \mathbf{B}^\gamma) = \mathbf{B}_\alpha \times (\mathbf{B}^\alpha \times \mathbf{B}_\delta).$$

Consequently, convoluting (3.17) and (3.18) with a^μ_δ we get the equalities such that their right-hand sides are linear combinations of $\mathbf{B}^\gamma, \dot{\mathbf{B}}^\gamma, \ddot{\mathbf{B}}^\gamma$, $\mathbf{B}^\alpha \times \mathbf{B}^\gamma, \dot{\mathbf{B}}^\alpha \times \mathbf{B}^\gamma$. And furthermore, the coefficients of these combinations are the functions of ω only. Consider first equality (3.17). The coefficients of $\mathbf{B}^\gamma, \dot{\mathbf{B}}^\gamma, \ddot{\mathbf{B}}^\gamma$ read as

$$\mathbf{B}^\gamma \quad : \quad a^\mu_\delta(\partial_\nu\partial^\nu)a_{\mu\gamma} - a^\mu_\delta\frac{\partial^2 a_{\nu\gamma}}{\partial x^\mu \partial x_\nu} = F_{\delta\gamma}(\omega); \tag{3.20}$$

$$\dot{\mathbf{B}}^\gamma \quad : \quad 2a^\mu_\delta\frac{\partial a_{\mu\gamma}}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} + g_{\delta\gamma}(\partial_\nu\partial^\nu)\omega - a^\mu_\delta\frac{\partial a_{\nu\gamma}}{\partial x_\nu}\frac{\partial \omega}{\partial x^\mu}$$

$$-a^\mu_\delta\frac{\partial a_{\nu\gamma}}{\partial x^\mu}\frac{\partial \omega}{\partial x_\nu} - a^\mu_\delta a_{\nu\gamma}\frac{\partial^2\omega}{\partial x_\nu \partial x^\mu} = G_{\delta\gamma}(\omega); \tag{3.21}$$

$$\ddot{\mathbf{B}}^\gamma \quad : \quad g_{\delta\gamma}\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - a^\mu_\delta a_{\nu\gamma}\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\mu} = H_{\delta\gamma}(\omega). \tag{3.22}$$

Turn now to coefficient (3.22). Convoluting the function $H_{\delta\gamma}(\omega)$ with the metric tensor $g^{\delta\gamma} = g_{\delta\gamma}$ yields that

$$g^{\delta\gamma}H_{\delta\gamma}(\omega) = 4\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - a_{\mu\delta}a^\delta_\nu\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x_\mu} = 4\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - g_{\mu\nu}\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x_\mu}$$
$$= 4\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} - \frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} = 3\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu}.$$

Hence we get that $\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu}$ is the function of ω only

$$\frac{\partial \omega}{\partial x_\nu}\frac{\partial \omega}{\partial x^\nu} = F_1(\omega). \tag{3.23}$$

Therefore,
$$a_\delta^\mu a_{\nu\gamma}\frac{\partial\omega}{\partial x_\nu}\frac{\partial\omega}{\partial x^\mu}=a_{\mu\delta}\frac{\partial\omega}{\partial x_\mu}a_{\nu\gamma}\frac{\partial\omega}{\partial x_\nu}=\tilde{H}_{\delta\gamma}(\omega),$$
whence
$$a_{\mu\delta}\frac{\partial\omega}{\partial x_\mu}=G_\delta(\omega). \tag{3.24}$$
In view of (3.23), (3.24) we get the equality
$$H_{\delta\gamma}(\omega)=g_{\delta\gamma}F_1-G_\delta G_\gamma.$$
Thus the function $H_{\delta\gamma}(\omega)$ coincides with $k_{\mu\gamma}$ from (3.15).

Convoluting (3.21) with the metric tensor $g^{\delta\gamma}$ gives
$$\begin{aligned}g^{\delta\gamma}G_{\delta\gamma}(\omega)&=2a_\delta^\mu\frac{\partial a_\mu^\delta}{\partial x_\nu}\frac{\partial\omega}{\partial x^\nu}+4(\partial_\nu\partial^\nu)\omega\\&-a_{\mu\delta}\frac{\partial\omega}{\partial x_\mu}\frac{\partial a_\nu^\delta}{\partial x_\nu}-a_{\mu\delta}\frac{\partial a_\nu^\delta}{\partial x_\nu}\frac{\partial\omega}{\partial x_\mu}-a_{\mu\delta}a_\nu^\delta\frac{\partial^2\omega}{\partial x_\mu\partial x_\nu}.\end{aligned} \tag{3.25}$$
Now using (3.11) we ensure that the relation
$$a_\delta^\mu\frac{\partial a_\mu^\delta}{\partial x_\nu}=\frac{1}{2}\frac{\partial}{\partial x_\nu}(a_\delta^\mu a_\mu^\delta)=\frac{1}{2}\frac{\partial}{\partial x_\nu}(g_\beta^\beta)=0, \tag{3.26}$$
as well as the relation
$$\frac{\partial}{\partial x_\nu}\left[a_{\mu\delta}a_\nu^\delta\frac{\partial\omega}{\partial x_\mu}\right]=\frac{\partial a_{\mu\delta}}{\partial x_\nu}a_\nu^\delta\frac{\partial\omega}{\partial x_\mu}+a_{\mu\delta}\frac{\partial a_\nu^\delta}{\partial x_\nu}\frac{\partial\omega}{\partial x_\mu}+a_{\mu\delta}a_\nu^\delta\frac{\partial^2\omega}{\partial x_\mu\partial x_\nu}$$
hold true. Owing to the fact that
$$\frac{\partial a_{\mu\delta}}{\partial x_\nu}a_\nu^\delta\frac{\partial\omega}{\partial x_\mu}=\frac{\partial a_\mu^\delta}{\partial x_\nu}a_{\nu\delta}\frac{\partial\omega}{\partial x_\mu},$$
we make sure that the relation
$$\frac{\partial a_{\mu\delta}}{\partial x_\nu}a_\nu^\delta\frac{\partial\omega}{\partial x_\mu}=a_{\mu\delta}\frac{\partial a_\nu^\delta}{\partial x_\mu}\frac{\partial\omega}{\partial x_\nu}$$
is valid, whence
$$\begin{aligned}a_{\mu\delta}\frac{\partial\omega}{\partial x_\mu}\frac{\partial a_\nu^\delta}{\partial x_\nu}+a_{\mu\delta}\frac{\partial a_\nu^\delta}{\partial x_\nu}\frac{\partial\omega}{\partial x_\nu}&=\frac{\partial}{\partial x_\nu}\left[a_{\mu\delta}a_\nu^\delta\frac{\partial\omega}{\partial x_\mu}\right]\\-a_{\mu\delta}a_\nu^\delta\frac{\partial^2\omega}{\partial x_\mu\partial x_\nu}&=\frac{\partial}{\partial x_\nu}\left[g_{\mu\nu}\frac{\partial\omega}{\partial x_\mu}\right]-g_{\mu\nu}\frac{\partial^2\omega}{\partial x_\mu\partial x_\nu}=0.\end{aligned} \tag{3.27}$$

With account of (3.25), (3.26) and (3.27) we get that

$$g^{\delta\gamma}G_{\delta\gamma}(\omega) = 4(\Box\omega - g_{\mu\nu}\frac{\partial^2\omega}{\partial x_\mu \partial x_\nu} = 3\Box\omega.$$

Consequently, the relation

$$\Box\omega = F_2(\omega) \tag{3.28}$$

holds.

Next, making sure that the equalities

$$\begin{aligned} a_{\mu\delta}\frac{\partial a_{\nu\gamma}}{\partial x_\mu}\frac{\partial\omega}{\partial x_\nu} + a_{\mu\delta}a_{\nu\gamma}\frac{\partial^2\omega}{\partial x_\nu \partial x_\mu} &= a_{\mu\delta}\frac{\partial}{\partial x_\mu}(a_{\nu\gamma}\frac{\partial\omega}{\partial x_\nu}) \\ = a_{\mu\delta}\frac{\partial}{\partial x_\nu}G_\gamma(\omega) = a_{\mu\delta}\dot{G}_\gamma(\omega)\frac{\partial\omega}{\partial x^\mu} &= \dot{G}_\gamma(\omega)G_\delta(\omega). \end{aligned}$$

hold and taking into account (3.21) yield

$$2a^\mu_\delta\frac{\partial a_{\mu\gamma}}{\partial x_\nu}\frac{\partial\omega}{\partial x^\nu} - G_\delta\frac{\partial a_{\nu\gamma}}{\partial x^\nu} = \tilde{G}_{\delta\gamma}(\omega). \tag{3.29}$$

Now, convoluting (3.29) with $g_{\mu\nu}$, we have

$$g_{\mu\nu}\tilde{G}_{\delta\gamma}(\omega) = g_{\mu\nu}G_\delta(\omega)\Big(2\frac{\partial a_{\mu\nu}}{\partial x_\mu} - g_{\mu\nu}\frac{\partial a_{\nu\gamma}}{\partial\partial x_\nu}\Big)$$

or, equivalently,

$$\frac{\partial a_{\mu\mu}}{\partial x_\mu} = H_\nu(\omega), \quad a^\mu_\delta\frac{\partial a_{\mu\gamma}}{\partial x_\nu}\frac{\partial\omega}{\partial x^\nu} = S_{\delta\gamma}(\omega). \tag{3.30}$$

With account of (3.24), (3.28), (3.30), we get that the coefficient of $\dot{\mathbf{B}}^\gamma$ in the reduced system (3.14) coincides with $l_{\mu\gamma}$ (3.15).

Finally, from the relation

$$a_{\mu\delta}\frac{\partial^2 a_{\mu\gamma}}{\partial x_\mu \partial x_\nu} = a_{\mu\delta}\frac{\partial}{\partial x_\nu}\Big(\frac{\partial a_{\nu\gamma}}{\partial x_\nu}\Big) = a_{\mu\delta}\frac{\partial}{\partial x_\mu}(H_\gamma(\omega)) = G_\delta(\omega)\dot{H}_\gamma(\omega),$$

with account of (3.20) it follows that

$$a^\mu_\delta\Box a_{\mu\gamma} = R_{\delta\gamma}(\omega).$$

Consequently, the function in the right-hand side of (3.20) coincides with $m_{\mu\gamma}$ from (3.15).

Analysis of (3.18) is carried out in the same way (we do not present here the corresponding calculations). The assertion is proved.

Thanks to the above assertion, the problem of symmetry reduction of the Yang-Mills equations by the subalgebras of the algebra $p(1,3)$ reduces to

routine substitution of the corresponding expressions for $a_{\mu\nu}, \omega$ into (3.16). We give below the final forms of the coefficients (3.15) of the reduced system of ordinary differential equations (3.14) for each of the subalgebras of the algebra $p(1,3)$:

$$
\begin{aligned}
L_1 \quad : \quad & k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \\
& g_{\mu\nu\gamma} = g_{\mu\gamma} d_\nu + g_{\nu\gamma} d_\mu - 2g_{\mu\nu} d_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
L_2 \quad : \quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \\
& g_{\mu\nu\gamma} = g_{\mu\gamma} a_\nu + g_{\nu\gamma} a_\mu - 2g_{\mu\nu} a_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
L_3 \quad : \quad & k_{\mu\gamma} = k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \\
& g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma, \quad h_{\mu\nu\gamma} = 0; \\
L_4 \quad : \quad & k_{\mu\gamma} = 4g_{\mu\gamma}\omega - a_\mu a_\gamma(\omega+1)^2 - d_\mu d_\gamma(\omega-1)^2 \\
& \quad -(a_\mu d_\gamma + a_\gamma d_\mu)(\omega^2 - 1), \\
& l_{\mu\gamma} = 4(g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma)) - 2k_\mu(a_\gamma - d_\gamma + k_\gamma\omega), \\
& m_{\mu\gamma} = 0, \\
& g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu\omega) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu\omega) \\
& \quad -2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma\omega)), \\
& h_{\mu\nu\gamma} = \frac{\epsilon}{2}[g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma] + \alpha\epsilon[(b_\mu c_\nu - c_\mu b_\nu)k_\gamma \\
& \quad +(b_\nu c_\gamma - c_\nu b_\gamma)k_\mu + (b_\gamma c_\mu - c_\gamma b_\mu)k_\nu]; \\
L_5 \quad : \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = -\epsilon c_\mu k_\gamma, \quad m_{\mu\gamma} = 0, \\
& g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \\
& h_{\mu\nu\gamma} = \frac{\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
L_6 \quad : \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 0, \\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu \\
& \quad +(a_\gamma d_\mu - a_\mu d_\gamma)b_\nu]; \\
L_7 \quad : \quad & k_{\mu\gamma} = -g_{\mu\gamma} - (b_\mu - \epsilon k_\mu e^\omega)(b_\gamma - \epsilon k_\gamma e^\omega), \\
& l_{\mu\gamma} = -2(a_\mu d_\gamma - a_\gamma d_\mu) + \epsilon e^\omega(b_\mu - \epsilon k_\mu e^\omega)k_\gamma, \\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(b_\nu - \epsilon k_\nu e^\omega) + g_{\nu\gamma}(b_\mu - \epsilon k_\mu e^\omega) \\
& \quad -2g_{\mu\nu}(b_\gamma - \epsilon k_\gamma e^\omega), \quad h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma \\
& \quad +(a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu]; \\
L_8 \quad : \quad & k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma), \\
& m_{\mu\gamma} = -\frac{1}{\omega}(\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) + b_\mu b_\gamma), \\
& g_{\mu\nu\gamma} = 2\sqrt{\omega}(g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma),
\end{aligned}
$$

$$
\begin{aligned}
& & h_{\mu\nu\gamma} = \frac{1}{2\sqrt{\omega}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma) + \frac{\alpha}{\sqrt{\omega}}((a_\mu d_\nu - a_\nu d_\mu)b_\gamma \\
& & \quad +(a_\nu d_\gamma - d_\nu a_\gamma)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu); \\
L_9 &:& k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0, \\
& & m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma, \\
& & g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma, \\
& & h_{\mu\nu\gamma} = a_\gamma(b_\mu c_\nu - c_\mu b_\nu) + a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) \\
& & \quad +a_\nu(b_\gamma c_\mu - c_\gamma b_\mu); \\
L_{10} &:& k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0, \\
& & m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma), \\
& & g_{\mu\nu\gamma} = g_{\mu\gamma}a_\nu + g_{\nu\gamma}a_\mu - 2g_{\mu\nu}a_\gamma, \\
& & h_{\mu\nu\gamma} = -[d_\gamma(b_\mu c_\nu - c_\mu b_\nu) + d_\mu(b_\nu c_\gamma - c_\nu b_\gamma) \\
& & \quad +d_\nu(b_\gamma c_\mu - c_\gamma b_\mu)]; \qquad (3.31) \\
L_{11} &:& k_{\mu\gamma} = -(a_\mu - d_\mu)(a_\gamma - d_\gamma), \quad l_{\mu\gamma} = 2(b_\mu c_\gamma - c_\mu b_\gamma), \\
& & m_{\mu\gamma} = 0, \\
& & g_{\mu\nu\gamma} = g_{\mu\gamma}(a_\nu - d_\nu) + g_{\nu\gamma}(a_\mu - d_\mu) - 2g_{\mu\nu}(a_\gamma - d_\gamma), \\
& & h_{\mu\nu\gamma} = \frac{1}{2}[(k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) \\
& & \quad +k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)]; \\
L_{12} &:& k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\omega^{-1}k_\mu k_\gamma, \quad m_{\mu\gamma} = -\alpha^2\omega^{-2}k_\mu k_\gamma, \\
& & g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \\
& & h_{\mu\nu\gamma} = \frac{1}{2}\omega^{-1}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \alpha\omega^{-1}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma \\
& & \quad +(k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu); \\
L_{13} &:& k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& & g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \\
& & h_{\mu\nu\gamma} = -((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu \\
& & \quad +(k_\gamma b_\mu - k_\mu b_\gamma)c_\nu); \\
L_{14} &:& k_{\mu\gamma} = -16(g_{\mu\gamma} + b_\mu b_\gamma), \quad l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0, \\
& & g_{\mu\nu\gamma} = 4(g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma), \\
L_{15} &:& k_{\mu\gamma} = -16[(1+\alpha^2)g_{\mu\gamma} + (c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma)], \\
& & l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0, \\
& & g_{\mu\nu\gamma} = -4[g_{\mu\gamma}(c_\nu - \alpha b_\nu) + g_{\nu\gamma}(c_\mu - \alpha b_\mu) \\
& & \quad -2g_{\mu\nu}(c_\gamma - \alpha b_\gamma)]; \\
L_{16} &:& k_{\mu\gamma} = -4\omega(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma) - 2\epsilon k_\gamma c_\mu\sqrt{\omega}, \\
& & m_{\mu\gamma} = -\omega^{-1}b_\mu b_\gamma, \quad g_{\mu\nu\gamma} = 2\sqrt{\omega}(g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma),
\end{aligned}
$$

$$
\begin{aligned}
& & h_{\mu\nu\gamma} &= \frac{1}{2}[\epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \frac{1}{\sqrt{\omega}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma)]; \\
L_{17} &: & k_{\mu\gamma} &= -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{2\omega+\alpha}{\omega(\omega+\alpha)+1}k_\mu k_\gamma, \\
& & m_{\mu\gamma} &= -4k_\mu k_\gamma(1+\omega(\alpha+\omega))^{-2}, \\
& & g_{\mu\nu\gamma} &= g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \\
& & h_{\mu\nu\gamma} &= \frac{1}{2}(\alpha+2\omega)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma)(1+\omega(\alpha+\omega))^{-1} \\
& & &\quad -2(1+\omega(\omega+\alpha))^{-1}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma \\
& & &\quad +(k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu); \\
L_{18} &: & k_{\mu\gamma} &= 4\omega g_{\mu\gamma} - (k_\mu\omega + a_\mu - d_\mu)(k_\gamma\omega + a_\gamma - d_\gamma), \\
& & l_{\mu\gamma} &= 6g_{\mu\gamma} + 4(a_\mu d_\gamma - a_\gamma d_\mu) - 3k_\gamma(k_\mu\omega + a_\mu - d_\mu), \\
& & m_{\mu\gamma} &= -k_\mu k_\gamma, \quad g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(k_\nu\omega + a_\nu - d_\nu) \\
& & &\quad +g_{\nu\gamma}(k_\mu\omega + a_\mu - d_\mu) - 2g_{\mu\nu}(k_\gamma\omega + a_\gamma - d_\gamma)), \\
& & h_{\mu\nu\gamma} &= \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
L_{19} &: & k_{\mu\gamma} &= -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu, \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& & g_{\mu\nu\gamma} &= g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma; \\
L_{20} &: & k_{\mu\gamma} &= -g_{\mu\gamma} - (c_\mu - \epsilon k_\mu)(c_\gamma - \epsilon k_\gamma), \\
& & l_{\mu\gamma} &= 2\epsilon k_\gamma c_\mu - 2k_\mu k_\gamma, \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& & g_{\mu\nu\gamma} &= g_{\mu\gamma}(\epsilon k_\nu - c_\nu) + g_{\nu\gamma}(\epsilon k_\mu - c_\mu) - 2g_{\mu\nu}(\epsilon k_\gamma - c_\gamma), \\
& & h_{\mu\nu\gamma} &= \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
L_{21} &: & k_{\mu\gamma} &= -g_{\mu\gamma} - (c_\mu - \alpha\epsilon k_\mu)(c_\gamma - \alpha\epsilon k_\gamma), \\
& & l_{\mu\gamma} &= 2(\epsilon k_\gamma c_\mu - \alpha k_\mu k_\gamma), \quad m_{\mu\gamma} = -k_\mu k_\gamma, \\
& & g_{\mu\nu\gamma} &= -g_{\mu\gamma}(c_\nu - \alpha\epsilon k_\nu) - g_{\nu\gamma}(c_\mu - \alpha\epsilon k_\mu) \\
& & &\quad +2g_{\mu\nu}(c_\gamma - \alpha\epsilon k_\gamma), \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
L_{22} &: & k_{\mu\gamma} &= -4\omega g_{\mu\gamma} - (a_\mu - d_\mu + k_\mu\omega)(a_\gamma - d_\gamma + k_\gamma\omega), \\
& & l_{\mu\gamma} &= 4[2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - a_\mu a_\gamma + d_\mu d_\gamma - \omega k_\mu k_\gamma], \\
& & m_{\mu\gamma} &= -2k_\mu k_\gamma, \quad g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu\omega) \\
& & &\quad +g_{\nu\gamma}(a_\mu - d_\mu + k_\mu\omega) - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma\omega)), \\
& & h_{\mu\nu\gamma} &= \frac{3\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) - \epsilon\alpha[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) \\
& & &\quad +(k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)].
\end{aligned}
$$

In the above formulae (3.31) we have $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$. Furthermore, α is an arbitrary parameter.

3.4 Exact solutions

Clearly, efficiency of the symmetry reduction procedure is subject to our ability to integrate the reduced systems of ordinary differential equations. Since the reduced equations are nonlinear, it is not at all clear that it will be possible to construct their particular or general solutions. That it why, we devote the first part of this subsection to describing our technique for integrating the reduced systems of nonlinear ordinary differential equations (the further details can be found in [33]).

Note that in contrast to the case of the nonlinear Dirac equation, it is not possible to construct the *general* solutions of the reduced systems (3.14)–(3.16). By this very reason, we give whenever possible their *particular* solutions, obtained by reduction of systems of equations in question by the number of components of the dependent function. Let us emphasize that the miraculous efficiency of the t'Hooft's ansatz [5] for the Yang-Mills equations is a consequence of the fact that it reduces the system of twelve differential equations to a single conformally-invariant wave equation.

Consider system (3.14)–(3.16), that corresponds to the subalgebra L_8. We adopt the following ansatz

$$\mathbf{B}_\mu = a_\mu \mathbf{e}_1 f(\omega) + d_\mu \mathbf{e}_2 g(\omega) + b_\mu \mathbf{e}_3 h(\omega) \tag{3.32}$$

for the vector-function $\mathbf{B}_\mu$, where $f(\omega),\ g(\omega),\ h(\omega)$ are new unknown smooth functions of ω and

$$\mathbf{e}_1 = (1,0,0)^T, \quad \mathbf{e}_2 = (0,1,0)^T, \quad \mathbf{e}_3 = (0,0,1)^T.$$

Now inserting (3.32) into (3.14), where the coefficients (3.15) are given in the list (3.31) for the case of the subalgebra L_8, we arrive at the system of relations

$$\begin{aligned}
&a_\mu \mathbf{e}_1[-4\omega\ddot{f} - 4\dot{f} - \frac{\alpha^2}{\omega}f + \frac{2\alpha e}{\sqrt{\omega}}gh + e^2(h^2+g^2)f] \\
&+d_\mu \mathbf{e}_2[-4\omega\ddot{g} - 4\dot{g} - \frac{\alpha^2}{\omega}g - \frac{2\alpha e}{\sqrt{\omega}}fh + e^2(h^2-f^2)g] \\
&+b_\mu \mathbf{e}_3[-4\omega\ddot{h} - 4\dot{h} + \omega^{-1}h - \frac{2\alpha e}{\sqrt{\omega}}fg + e^2(g^2-f^2)h] = 0.
\end{aligned}$$

It is equivalent to the following system of three ordinary differential equations:

$$\begin{aligned}
&4\omega\ddot{f} + 4\dot{f} + \frac{\alpha^2}{\omega}f - \frac{2\alpha e}{\sqrt{\omega}}g\,h - e^2(h^2+g^2)f = 0, \\
&4\omega\ddot{g} + 4\dot{g} + \frac{\alpha^2}{\omega}g + \frac{2\alpha e}{\sqrt{\omega}}f\,h - e^2(h^2-f^2)g = 0, \\
&4\omega\ddot{h} + 4\dot{h} - \omega^{-1}h + \frac{2\alpha e}{\sqrt{\omega}}f\,g - e^2(g^2-f^2)h = 0.
\end{aligned} \tag{3.33}$$

So that we reduce system of twelve ordinary differential equations (3.14) to the one containing three equations only.

Next, choosing

$$\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega) \tag{3.34}$$

and inserting this expression into (3.14) with coefficients given by formulae (3.31) for the case of the subalgebra L_8 under $\alpha = 0$ yield the system of two ordinary differential equations

$$4\omega\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4\omega\ddot{g} + 4\dot{g} - \omega^{-1} g = 0.$$

Note that the second equation of the above system is linear.

In a similar way we have reduced some other systems of ordinary differential equations (3.14) to systems of two or three equations. Below we list the substitutions for $\mathbf{B}_\mu(\omega)$ and corresponding systems of ordinary differential equations. Numbering of the systems below corresponds to numbering of the corresponding subalgebras L_j of the algebra $p(1,3)$.

$$\begin{array}{ll}
1. & \mathbf{B}_\mu = a_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega) + c_\mu \mathbf{e}_3 h(\omega), \\
& \ddot{f} - e^2(g^2 + h^2) f = 0, \quad \ddot{g} + e^2(f^2 - h^2) g = 0, \\
& \ddot{h} + e^2(f^2 - g^2) h = 0. \\
2. & \mathbf{B}_\mu = b_\mu \mathbf{e}_1 f(\omega) + c_\mu \mathbf{e}_2 g(\omega) + d_\mu \mathbf{e}_3 h(\omega), \\
& \ddot{f} + e^2(g^2 + h^2) f = 0, \quad \ddot{g} + e^2(f^2 + h^2) g = 0, \\
& \ddot{h} + e^2(f^2 + g^2) h = 0. \\
5. & \mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega), \\
& \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
8.1. & (\alpha = 0) \;\; \mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega), \\
& 4\omega\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4\omega\ddot{g} + 4\dot{g} - \omega^{-1} g = 0. \\
8.2. & \mathbf{B}_\mu = a_\mu \mathbf{e}_1 f(\omega) + d_\mu \mathbf{e}_2 g(\omega) + b_\mu \mathbf{e}_3 h(\omega), \\
& 4\omega\ddot{f} + 4\dot{f} - \dfrac{\alpha^2}{\omega} f - \dfrac{2\alpha e}{\sqrt{\omega}} g h - e^2(h^2 + g^2) f = 0, \\
& 4\omega\ddot{g} + 4\dot{g} + \dfrac{\alpha^2}{\omega} g + \dfrac{2\alpha e}{\sqrt{\omega}} f h + e^2(f^2 - h^2) g = 0, \\
& 4\omega\ddot{h} + 4\dot{h} - \omega^{-1} h + \dfrac{2\alpha e}{\sqrt{\omega}} f g + e^2(f^2 - g^2) h = 0. \\
14.1. & \mathbf{B}_\mu = a_\mu \mathbf{e}_1 f(\omega) + d_\mu \mathbf{e}_2 g(\omega) + c_\mu \mathbf{e}_3 h(\omega), \\
& 16\ddot{f} - e^2(h^2 + g^2) f = 0, \quad 16\ddot{g} + e^2(f^2 - h^2) g = 0, \\
& 16\ddot{h} + e^2(f^2 - g^2) h = 0. \\
14.2. & \mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + c_\mu \mathbf{e}_2 g(\omega), \\
& 16\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0. \\
15.1. & \mathbf{B}_\mu = a_\mu \mathbf{e}_1 f(\omega) + d_\mu \mathbf{e}_2 g(\omega) + (1 + \alpha^2)^{-\frac{1}{2}} (\alpha c_\mu + b_\mu) \mathbf{e}_3 h(\omega),
\end{array} \tag{3.35}$$

$$
\begin{aligned}
&16(1+\alpha^2)\ddot{f} - e^2(h^2+g^2)f = 0,\\
&16(1+\alpha^2)\ddot{g} + e^2(f^2-h^2)g = 0,\\
&16(1+\alpha^2)\ddot{h} + e^2(f^2-g^2)h = 0.
\end{aligned}
$$

15.2. $\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + (1+\alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\mathbf{e}_2 g(\omega)$,
$16(1+\alpha^2)\ddot{f} - e^2 f g^2 = 0, \quad \ddot{g} = 0.$

16. $\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega)$,
$4\omega\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4\omega\ddot{g} + 4\dot{g} - \omega^{-1} g = 0.$

18. $\mathbf{B}_\mu = b_\mu \mathbf{e}_1 f(\omega) + c_\mu \mathbf{e}_2 g(\omega)$,
$4\omega\ddot{f} + 6\dot{f} + e^2 g^2 f = 0, \quad 4\omega\ddot{g} + 6\dot{g} + e^2 f^2 g = 0.$

19. $\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$

20. $\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$

21. $\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$

22. $(\alpha = 0) \quad \mathbf{B}_\mu = b_\mu \mathbf{e}_1 f(\omega) + c_\mu \mathbf{e}_2 g(\omega)$,
$4\omega\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \quad 4\omega\ddot{g} + 8\dot{g} + e^2 f^2 g = 0.$

So, combining symmetry reduction by the number of independent variables and direct reduction by the number of the components of the function to be found we have reduced the $SU(2)$ Yang-Mills equations (3.1) to comparatively simple systems of ordinary differential equations (3.35).

As a next step, we briefly review the procedure of integration of equations (3.35).

Choosing $f = 0,\ g = h = u(\omega)$ reduces system 1 to the equation

$$\ddot{u} = e^2 u^3, \tag{3.36}$$

that is integrated in terms of the elliptic functions. Note, that this equation has the solution that is expressed in terms of elementary functions, namely,

$$u = \sqrt{2}(e\omega - C)^{-1}, \quad C \in \mathbf{R}.$$

System 2 with $f = g = h = u(\omega)$ reduces to the equation

$$\ddot{u} + 2e^2 u^3 = 0,$$

that is also integrated in terms of the elliptic functions.

Upon integrating the second equation of system 5 we get

$$g = C_1\omega + C_2, \quad C_1, C_2 \in \mathbf{R}.$$

Provided $C_1 \neq 0$, the constant C_2 is negligible and we may put $C_2 = 0$. With this condition the first equation of system 5 reads

$$\ddot{f} - e^2 C_1^2 \omega^2 f = 0. \tag{3.37}$$

The general solution of equation (3.37), which is equivalent to the Bessel equation, is given by the formula

$$f = \sqrt{\omega} Z_{\frac{1}{4}}(\frac{i\,e}{2} C_1 \omega^2).$$

Here we use the designations $Z_\nu(\omega) = C_3 J_\nu(\omega) + C_4 Y_\nu(\omega)$, where J_ν, Y_ν are the Bessel functions and C_3, C_4 are arbitrary constants.

Given the condition $C_1 = 0,\ C_2 \neq 0$, the general solution of the first equation of system 5 reads as

$$f = C_3 \cosh(C_2 e\,\omega) + C_4 \sinh(C_2 e\,\omega),$$

where $C_3,\ C_4$ are arbitrary real constants.

Finally, if $C_1 = C_2 = 0$, then the general solution of the first equation of system 5 is given by formula $f = C_3\omega + C_4,\ C_3,\ C_4 \in \mathbf{R}$.

Next, we integrate the second equation of system 8.1 to obtain

$$g = C_1\sqrt{\omega} + C_2(\sqrt{\omega})^{-1},$$

where $C_1,\ C_2$ are arbitrary integration constants. Inserting the function g into the first equation of system 8.1 yields the linear differential equation

$$4\omega^2 \ddot{f} + 4\omega \dot{f} - e^2(C_1\omega + C_2)^2 f = 0. \tag{3.38}$$

For the case $C_1C_2 \neq 0$, equation (3.38) is related to the Whitteker equation. Here we restrict our considerations to the case $C_1C_2 = 0$, thus getting

$$\begin{aligned}
&a)\quad C_1 \neq 0, \quad C_2 = 0, \quad f = Z_0\Big[\frac{i\,e}{2} C_1 \omega\Big];\\
&b)\quad C_1 = 0, \quad C_2 \neq 0, \quad f = C_3 \omega^{\frac{eC_2}{2}} + C_4 \omega^{-\frac{eC_2}{2}};\\
&c)\quad C_1 = C_2 = 0, \quad f = C_3 \ln\omega + C_4,
\end{aligned}$$

where $C_3,\ C_4$ are arbitrary integration constants.

Analyzing equations 14.1 and 14.2 we arrive at the conclusion that they reduce to equations 1 and 5, correspondingly, if we replace e by $\frac{e}{4}$. Analogously, replacing in systems 1, 5 the parameter e by $\frac{e}{4}(1+\alpha^2)^{-\frac{1}{2}}$ yields systems 15.1 and 15.2, respectively.

Finally, system 22 with $\alpha = 0$ is reduced by the change of the dependent variable $f = g = u(\omega)$ to the Emden-Fauler equation

$$\omega\ddot{u} + 2\dot{u} + \frac{e^2}{4} u^3 = 0,$$

that has the following particular solution $u = e^{-1}\omega^{-\frac{1}{2}}$.

We have not succeeded in integrating systems of ordinary differential equations 8.2 and 18. Furthermore, systems 19, 20, 21 coincide with system 5 and system 16 coincides with system 8.1.

Inserting the obtained forms of the functions f, g, h into (3.35) with the subsequent substitution of the latter expression into the corresponding ansatz (3.8)–(3.10) yield invariant solutions of the $SU(2)$ Yang-Mills equations (3.1). Note that solutions of systems 5, 8.1, 14.2, 15.2, 16, 19, 20, 21 with $g = 0$, give rise to Abelian solutions of the Yang-Mills equation, i.e., to solutions satisfying the additional restriction $\mathbf{A}_\mu \times \mathbf{A}_\nu = \mathbf{0}$. Such solutions are of low interest for physical applications and are not considered here. Below we give the full list of non-Abelian invariant solutions of equations (3.1)

$$
\begin{aligned}
&1.\quad \mathbf{A}_\mu = (\mathbf{e}_2 b_\mu + \mathbf{e}_3 c_\mu)\sqrt{2}(edx - \lambda)^{-1};\\
&2.\quad \mathbf{A}_\mu = (\mathbf{e}_2 b_\mu + \mathbf{e}_3 c_\mu)[\lambda \mathrm{sn}\,(\frac{\sqrt{2}}{2}e\lambda dx)\mathrm{dn}\,(\frac{\sqrt{2}}{2}e\lambda dx)][\mathrm{cn}\,\frac{\sqrt{2}}{2}e\lambda dx)]^{-1};\\
&3.\quad \mathbf{A}_\mu = (\mathbf{e}_2 b_\mu + \mathbf{e}_3 c_\mu)\lambda[\mathrm{cn}\,(e\lambda dx)]^{-1};\\
&4.\quad \mathbf{A}_\mu = (\mathbf{e}_1 b_\mu + \mathbf{e}_2 c_\mu + \mathbf{e}_3 c_\mu)\lambda \mathrm{cn}\,(e\lambda ax);\\
&5.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}[\frac{i}{2}e\lambda(cx)^2] + \mathbf{e}_2 b_\mu \lambda cx;\\
&6.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(e\lambda cx) + \lambda_2\sinh(e\lambda cx)] + \mathbf{e}_2 b_\mu \lambda;\\
&7.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu Z_0[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)] + \mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda;\\
&8.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu[\lambda_1((bx)^2 + (cx)^2))^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}\\
&\qquad +\mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1};\\
&9.\quad \mathbf{A}_\mu = [\mathbf{e}_2(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx) + \mathbf{e}_3 c_\mu]\lambda \mathrm{sn}\,(\frac{e\sqrt{2}}{8}\lambda(4bx\\
&\qquad +(kx)^2))\mathrm{dn}\,(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2))(\mathrm{cn}\,(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)))^{-1};\\
&10.\quad \mathbf{A}_\mu = [\mathbf{e}_2(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx) + \mathbf{e}_3 c_\mu]\\
&\qquad \times\lambda[\mathrm{cn}\,(\frac{e\sqrt{2}\lambda}{8}(4bx + (kx)^2))]^{-1};\\
&11.\quad \mathbf{A}_\mu = [\mathbf{e}_2(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx) + \mathbf{e}_3 c_\mu]\\
&\qquad \times 4\sqrt{2}(e(4bx + (kx)^2) - \lambda)^{-1};\\
&12.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu\sqrt{4bx + (kx)^2}Z_{\frac{1}{4}}(\frac{ie\lambda}{8}(4bx + (kx)^2)^2)\\
&\qquad +\mathbf{e}_2 c_\mu\lambda(4bx + (kx)^2);\\
&13.\quad \mathbf{A}_\mu = \mathbf{e}_1 k_\mu(\lambda\cosh(\frac{e\lambda}{4}(4bx + (kx)^2))
\end{aligned}
\tag{3.39}
$$

$$+\lambda_2 \sinh(\frac{e\lambda}{4}(4bx+(kx)^2))) + \mathbf{e}_2 c_\mu \lambda;$$

14. $$\mathbf{A}_\mu = \{\mathbf{e}_2(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx)$$
$$+\mathbf{e}_3(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx)(1+\alpha^2)^{-\frac{1}{2}}\}$$
$$\times \lambda \mathrm{sn}\,[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}]$$
$$\times \mathrm{dn}\,[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}]$$
$$\times \{\mathrm{cn}\,[\frac{e\lambda\sqrt{2}}{8}((4\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}]\}^{-1};$$

15. $$\mathbf{A}_\mu = \{\mathbf{e}_2(d_\mu - \frac{1}{8}k_\mu(kx)^2) - \frac{1}{2}b_\mu kx)$$
$$+\mathbf{e}_3(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx)(1+\alpha^2)^{-\frac{1}{2}}\}$$
$$\times \{\mathrm{cn}\,[\frac{e\lambda}{4}(4\alpha bx - cx) + \alpha(kx)^2(1+\alpha^2)^{-\frac{1}{2}}]\}^{-1};$$

16. $$\mathbf{A}_\mu = \{\mathbf{e}_2(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx)$$
$$+\mathbf{e}_3(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx)(1+\alpha^2)^{-\frac{1}{2}}\}$$
$$\times 4\sqrt{2}(1+\alpha^2)^{\frac{1}{2}}[e(4(\alpha bx - cx) + \alpha(kx)^2)]^{-1};$$

17. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu\{\sqrt{4(\alpha bx - cx) + \alpha(kx)^2}\, Z_{\frac{1}{4}}(\frac{ie\lambda}{8}(4(\alpha bx - cx)$$
$$+\alpha(kx)^2)^2)(1+\alpha^2)^{-\frac{1}{2}}\}$$
$$+\mathbf{e}_2(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx)\lambda(4(\alpha bx - cx) + \alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}};$$

18. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu\{\mathrm{cn}\,[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha(kx)^2)]$$
$$+\lambda_2 \sinh[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha(kx)^2]\}$$
$$+\mathbf{e}_2(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx)\lambda(1+\alpha^2)^{-\frac{1}{2}};$$

19. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1} Z_0[\frac{ie\lambda}{2}((bx)^2 + (cx)^2)] + \mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda;$$

20. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}]$$
$$+\mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1};$$

21. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}\sqrt{cx}\, Z_{\frac{1}{4}}(\frac{ie\lambda}{2}(cx)^2) + \mathbf{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda cx;$$

22. $$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}[\lambda_1 \cosh(\lambda ecx) + \lambda_2 \sinh(\lambda ecx)]$$

$$+\mathbf{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda;$$

23. $\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}\sqrt{\ln|kx| - cx}\, Z_{\frac{1}{4}}(\frac{ie\lambda}{2}(\ln|kx| - cx)^2)$
$+\mathbf{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda(\ln|kx| - cx);$

24. $\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}[\lambda_1 \cosh(\lambda e(\ln|kx| - cx)) + \lambda_2 \sinh(\lambda e(\ln|kx| - cx))]$
$+\mathbf{e}_2[b_\mu - k_\mu bx(kx)^{-1}]\lambda;$

25. $\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}\sqrt{\alpha\ln|kx| - cx}\, Z_{\frac{1}{4}}(\frac{ie\lambda}{2}(\alpha\ln|kx| - cx)^2)$
$+\mathbf{e}_2(b_\mu - k_\mu bx - \ln|kx|)(kx)^{-1})\lambda(\alpha\ln|kx| - cx);$

26. $\mathbf{A}_\mu = \mathbf{e}_1 k_\mu |kx|^{-1}[\lambda_1 \cosh(\lambda e(\alpha\ln|kx| - cx))$
$+\lambda_2 \sinh(\lambda e(\alpha\ln|kx| - cx))]$
$+\mathbf{e}_2(b_\mu - k_\mu(bx - \ln|kx|^{-1})(kx)^{-1})\lambda;$

27. $\mathbf{A}_\mu = \{\mathbf{e}_1(b_\mu - k_\mu bx(kx)^{-1})$
$+\mathbf{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}e^{-1}(x_\nu x^\nu)^{-\frac{1}{2}};$

28. $\mathbf{A}_\mu = \{\mathbf{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \mathbf{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}f(x_\nu x^\nu).$

In the above formulae the symbol $Z_\alpha(\omega)$ stands for the Bessel function, $\mathrm{sn}\,(\omega), \mathrm{dn}\,(\omega), \mathrm{cn}\,(\omega)$ are the Jacobi elliptic functions having the module $\frac{\sqrt{2}}{2}$; $f(x_\nu x^\nu)$ is the general solution of the ordinary differential equation

$$\omega^2 \ddot{f} + 2\omega\dot{f} + \frac{e^2}{4}f^3 = 0$$

and $\lambda, \lambda_1, \lambda_2$ are arbitrary real constants.

4 Conditional symmetry and new solutions of the Yang-Mills equations.

With all the wealth of exact solutions obtainable through Lie symmetries of the Yang-Mills equations, it is possible to construct solutions, that cannot be derived by the symmetry reduction method. The source of these solutions is *conditional* or *non-classical* symmetry of the Yang-Mills equations.

The first paper devoted to non-classical symmetry of partial differential equations was published by Bluman and Cole [57]. However, the real importance of these symmetries was understood much later after appearing the papers [58]–[61], [31, 32], where the method of conditional symmetries had been used in order to construct new exact solutions of a number of nonlinear partial differential equations.

The methods for dimensional reduction of partial differential equations based on their conditional symmetry can be conventionally classified into two

principal groups. The first group is formed by the direct methods (the ansatz method by Fushchych and the direct method by Clarkson & Kruskal), relying upon a special *ad-hoc* representation of the solution to be found in the form of the ansatz containing some arbitrary elements (functions) $f_1, f_2, \ldots, f_n$ and unknown functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ with fewer number of dependent variables. Inserting the ansatz in question into equation under study and requiring for the obtained relation to be equivalent to a system of partial differential equations for the functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ yield nonlinear determining equations for the functions $f_1, f_2, \ldots, f_n$. Having solved the latter yields a number of ansatzes reducing a given partial differential equation to one having fewer number of dependent variables. The second group of methods (the non-classical method by Bluman & Cole, the method of conditional symmetries by Fushchych and the method of side conditions by Olver & Rosenau) may be regarded as infinitesimal ones. They are in line with the traditional Lie approach to the reduction of partial differential equations, since they exploit symmetry properties of the equation under study in order to construct its invariant solutions. And again any deviation from the standard Lie approach requires solving over-determined system of nonlinear determining equations. A more profound analysis of similarities and differences between these approaches can be found in [33, 56, 64].

So the principal idea of the method of ansatzes, as well as, of the direct method of reduction of partial differential equations is a special choice of the class of functions to which a solution to be found should belong. Within the framework of the above methods a solution of system (3.1) is looked for in the form

$$\mathbf{A}_\mu = H_\mu\Big(x, \mathbf{B}_\nu(\omega(x))\Big), \quad \mu = 0, 1, 2, 3,$$

where H_μ are smooth functions chosen in such a way that substitution of the above expressions into the Yang-Mills equations yields a system of ordinary differential equations for new unknown vector-functions $\mathbf{B}_\nu$ of one variable ω. However being posed in this way, the problem of reduction of the Yang-Mills equations seems to be hopeless. Indeed, even if we restrict ourselves to the case of a linear dependence of the above ansatz on B_ν

$$\mathbf{A}_\mu(x) = R_{\mu\nu}(x)\mathbf{B}^\nu(\omega), \tag{4.1}$$

where $\mathbf{B}_\nu(\omega)$ are new unknown vector-functions and $\omega = \omega(x)$ is the new independent variable, then the requirement of reduction of (3.1) to a system of ordinary differential equations by virtue of (4.1) gives rise to the system of nonlinear partial differential equations for 17 unknown functions $R_{\mu\nu}$, ω. And what is more, the system obtained is not at all simpler than the initial Yang-Mills equations (3.1). Consequently, an additional information about the structure of the matrix-function $R_{\mu\nu}$ should be input into ansatz (4.1). This can be done in various ways. But the most natural one is to use the

information about the structure of solutions provided by the Lie symmetry of the equation under study.

In [33] we suggest an effective approach to study of conditional symmetry of the nonlinear Dirac equation based on its Lie symmetry. We have observed that all the Poincaré-invariant ansatzes for the Dirac field $\psi(x)$ can be represented in the unified form by introducing several arbitrary elements (functions) $u_1(x),\ u_2(x),\ldots,u_N(x)$. As a result, we get an ansatz for the field $\psi(x)$ which reduces the nonlinear Dirac equation to system of ordinary differential equations, provided functions $u_i(x)$ satisfy some compatible over-determined system of nonlinear partial differential equations. After integrating it we have obtained a number of new ansatzes that cannot in principle be obtained within the framework of the classical Lie approach.

Here, following [49] we will show that the same idea proves to work efficiently for obtaining new (non-Lie) reductions of the Yang-Mills equations and for constructing new exact solutions of system (3.1).

4.1 Non-classical reductions of the Yang-Mills equations

In the previous section we give the complete list of $P(1{,}3)$-inequivalent ansatzes for the Yang-Mills field, which are invariant under the three-parameter subgroups of the Poincaré group $P(1,3)$. These ansatzes can be represented in the unified form (3.8), where $\mathbf{B}_\nu(\omega)$ are new unknown vector-functions, $\omega=\omega(x)$ is the new independent variable and the functions $a_{\mu\nu}(x)$ are given by (3.9).

In (3.9), $\theta_\mu(x)$ are some smooth functions, and what is more $\theta_a=\theta_a(\xi,$ $b_\mu x^\mu,\ c_\mu x^\mu)$, $a=1,2$; $\xi=(1/2)k_\mu x^\mu=(1/2)(a_\mu x^\mu+d_\mu x^\mu)$; $a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ are arbitrary constants satisfying relations (3.5).

The choice of the functions $\omega(x),\ \theta_\mu(x)$ is determined by the requirement that substitution of ansatz (3.8) into the Yang-Mills equations yields a system of ordinary differential equations for the vector function $\mathbf{B}_\mu(\omega)$. By the direct check one can become convinced of the validity of the following statement [33, 49]:

Assertion 4.1 *Ansatz (3.8), (3.9) reduces the Yang-Mills equations (3.1) to system of ordinary differential equations, if and only if, the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the following system of partial differential equations:*

$$
\begin{aligned}
&1)\quad \omega_{x_\mu}\omega_{x^\mu}=F_1(\omega),\\
&2)\quad \Box\omega=F_2(\omega),\\
&3)\quad a_{\alpha\mu}\omega_{x_\alpha}=G_\mu(\omega),\\
&4)\quad a_{\alpha\mu x_\alpha}=H_\mu(\omega),\\
&5)\quad R^\alpha_\mu a_{\alpha\nu x_\beta}\omega_{x^\beta}=Q_{\mu\nu}(\omega),
\end{aligned}
\tag{4.2}
$$

6) $R^{\alpha}_{\mu}\Box a_{\alpha\nu} = S_{\mu\nu}(\omega)$,

7) $R^{\alpha}_{\mu} a_{\alpha\nu x_\beta} a_{\beta\gamma} + a^{\alpha}_{\nu} a_{\alpha\gamma x_\beta} a_{\beta\mu} + a^{\alpha}_{\gamma} a_{\alpha\mu x_\beta} a_{\beta\nu} = T_{\mu\nu\gamma}(\omega)$,

where $F_1,\ F_2,\ G_\mu, \ldots, T_{\mu\nu\gamma}$ *are some smooth functions of* $\omega,\ \mu,\ \nu,\ \gamma = 0, 1, 2, 3$. *And what is more, the reduced system has the form*

$$\begin{aligned} & k_{\mu\gamma}\ddot{\mathbf{B}}^{\gamma} + l_{\mu\gamma}\dot{\mathbf{B}}^{\gamma} + m_{\mu\gamma}\mathbf{B}^{\gamma} + eq_{\mu\nu\gamma}\dot{\mathbf{B}}^{\nu} \times \mathbf{B}^{\gamma} + eh_{\mu\nu\gamma}\mathbf{B}^{\nu} \times \mathbf{B}^{\gamma} \\ & \quad + e^2\mathbf{B}_{\gamma} \times (\mathbf{B}^{\gamma} \times \mathbf{B}_{\mu}) = \mathbf{0}, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} k_{\mu\gamma} &= g_{\mu\gamma}F_1 - G_\mu G_\gamma, \\ l_{\mu\gamma} &= g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ m_{\mu\gamma} &= S_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\ q_{\mu\nu\gamma} &= g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ h_{\mu\nu\gamma} &= (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}. \end{aligned} \tag{4.4}$$

Consequently, to describe all the ansatzes of the form (3.8), (3.9) reducing the Yang-Mills equations to a system of ordinary differential equations one has to construct the general solution of the over-determined system of partial differential equations (3.9), (4.2). Let us emphasize that system (3.9), (4.2) is compatible since the ansatzes for the Yang-Mills field $\mathbf{A}_\mu(x)$ invariant under the three-parameter subgroups of the Poincaré group satisfy equations (3.9), (4.2) with some specific choice of the functions $F_1,\ F_2, \ldots, T_{\mu\nu\gamma}$ [35].

Integration of system of nonlinear partial differential equations (3.9), (4.2) has been performed in [33, 49]. Here we indicate the principal steps of the integration procedure. While integrating (3.9), (4.2) we use essentially the fact that the general solution of system of equations 1, 2 from (4.2) is known [62]. With already known $\omega(x)$ in hand we proceed to integrating linear partial differential equations 3, 4 from (4.2). Next, we insert the results obtained into the remaining equations and get the final forms of the functions $\omega(x)$, $\theta_\mu(x)$.

Before presenting the results of integration of system of partial differential equations (3.9), (4.2) we make the following remark. As the direct check shows, the structure of ansatz (3.8), (3.9) is not altered by the change of variables

$$\begin{aligned} & \omega \to \omega' = T(\omega), \quad \theta_0 \to \theta_0' = \theta_0 + T_0(\omega), \\ & \theta_1 \to \theta_1' = \theta_1 + e^{\theta_0}\Big(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3\Big), \\ & \theta_2 \to \theta_2' = \theta_2 + e^{\theta_0}\Big(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3\Big), \\ & \theta_3 \to \theta_3' = \theta_3 + T_3(\omega), \end{aligned} \tag{4.5}$$

where $T(\omega),\ T_\mu(\omega)$ are arbitrary smooth functions. That is why, solutions of system (3.9), (4.2) connected by relations (4.5) are considered as equivalent.

Integrating the system of partial differential equations under study within the above equivalence relations we obtain the set of ansatzes containing the ones equivalent to the Poincaré-invariant ansatzes, obtained in the previous section. That is why, we concentrate on essentially new (non-Lie) ansatzes. It so happens that our approach gives rise to non-Lie ansatzes, provided the functions $\omega(x),\ \theta_\mu(x)$ within the equivalence relations (4.5) have the form

$$\theta_\mu = \theta_\mu(\xi,\, bx,\, cx), \quad \omega = \omega(\xi,\, bx,\, cx), \tag{4.6}$$

where, as earlier, $bx = b_\mu x^\mu, cx = c_\mu x^\mu$.

List of inequivalent solutions of system of partial differential equations (3.9), (4.2) satisfying (4.6) is exhausted by the following solutions:

$$\begin{aligned}
&1)\quad \theta_0 = \theta_3 = 0, \quad \omega = (1/2)kx, \quad \theta_1 = w_0(\xi)bx + w_1(\xi)cx,\\
& \theta_2 = w_2(\xi)bx + w_3(\xi)cx;\\
&2)\quad \omega = bx + w_1(\xi), \quad \theta_0 = \alpha\Big(cx + w_2(\xi)\Big),\\
& \theta_a = -(1/4)\dot{w}_a(\xi), \quad a = 1, 2, \quad \theta_3 = 0,\\
&3)\quad \theta_0 = T(\xi), \quad \theta_3 = w_1(\xi), \quad \omega = bx\cos w_1 + cx\sin w_1 + w_2(\xi),\\
& \theta_1 = \Big((1/4)(\varepsilon e^T + \dot{T})(bx\sin w_1 - cx\cos w_1) + w_3(\xi)\Big)\sin w_1\\
& +(1/4)\Big(\dot{w}_1(bx\sin w_1 - cx\cos w_1) - \dot{w}_2\Big)\cos w_1,\\
& \theta_2 = -\Big((1/4)(\varepsilon e^T + \dot{T})(bx\sin w_1 - cx\cos w_1) + w_3(\xi)\Big)\cos w_1\\
& +(1/4)\Big(\dot{w}_1(bx\sin w_1 - cx\cos w_1) - \dot{w}_2\Big)\sin w_1;\\
&4)\quad \theta_0 = 0, \quad \theta_3 = \arctan\Big([cx + w_2(\xi)][bx + w_1(\xi)]^{-1}\Big),\\
& \theta_a = -(1/4)\dot{w}_a(\xi), \quad a = 1, 2,\\
& \omega = \Big([bx + w_1(\xi)]^2 + [cx + w_2(\xi)]^2\Big)^{1/2}.
\end{aligned} \tag{4.7}$$

Here $\alpha \neq 0$ is an arbitrary constant, $\varepsilon = \pm 1$, $w_0,\ w_1,\ w_2,\ w_3$ are arbitrary smooth functions on $\xi = (1/2)kx$, $T = T(\xi)$ is a solution of the nonlinear ordinary differential equation

$$(\dot{T} + \varepsilon e^T)^2 + \dot{w}_1^2 = \varkappa e^{2T}, \quad \varkappa \in \mathbb{R}^1, \tag{4.8}$$

a dot over the symbol denotes differentiation with respect to ξ.

Inserting ansatz (3.8), where $a_{\mu\nu}(x)$ are given by formulae (3.9), (4.7), into the Yang-Mills equations yields systems of nonlinear ordinary differential equations of the form (4.3), where

$$1)\quad k_{\mu\gamma} = -(1/4)k_\mu k_\gamma, \quad l_{\mu\gamma} = -(w_0 + w_3)k_\mu k_\gamma,$$

$$
\begin{aligned}
&m_{\mu\gamma} = -4\ (w_0^2 + w_1^2 + w_2^2 + w_3^2)k_\mu k_\gamma - (\dot{w}_0 + \dot{w}_3)k_\mu k_\gamma,\\
&q_{\mu\nu\gamma} = (1/2)(g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma),\\
&h_{\mu\nu\gamma} = (w_0 + w_3)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + 2(w_1 - w_2)\Big((k_\mu b_\nu - k_\nu b_\mu)\ c_\gamma\\
&\quad +(b_\mu c_\nu - b_\nu c_\mu)k_\gamma + (c_\mu k_\nu - c_\nu k_\mu)b_\gamma\Big);
\end{aligned}
$$

$$
\begin{aligned}
2)\quad &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma),\\
&q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,\\
&h_{\mu\nu\gamma} = \alpha\Big((a_\mu d_\nu - a_\nu d_\mu)c_\gamma + (d_\mu c_\nu - d_\nu c_\mu)a_\gamma + (c_\mu a_\nu - c_\nu a_\mu)d_\gamma\Big);
\end{aligned}
$$

$$
\begin{aligned}
3)\quad &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -(\varepsilon/2)b_\mu k_\gamma,\\
&m_{\mu\gamma} = -(\varkappa/4)k_\mu k_\gamma, \quad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,\\
&h_{\mu\nu\gamma} = (\varepsilon/4)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
4)\quad &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\omega^{-1}(g_{\mu\gamma} + b_\mu b_\gamma),\\
&m_{\mu\gamma} = -\omega^{-2}c_\mu c_\gamma, \quad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma,\\
&h_{\mu\nu\gamma} = (1/2)\omega^{-1}(g_{\mu\gamma}b_\nu - g_{\mu\nu}b_\gamma).
\end{aligned}
$$

4.2 Exact solutions

Systems (4.3), (4.9) contain twelve nonlinear second-order ordinary differential equations with variable coefficients. That is why, there is little hope to construct their general solutions. Nevertheless, it is possible to obtain particular solutions of system (4.3), whose coefficients are given by formulae 2–4 from (4.9).

Consider, as an example, system of ordinary differential equations (4.3) with coefficients given by the formulae 2 from (4.9). We look for its solutions of the form

$$\mathbf{B}_\mu = k_\mu \mathbf{e}_1 f(\omega) + b_\mu \mathbf{e}_2 g(\omega), \quad fg \neq 0, \tag{4.10}$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$.

Substituting expression (4.10) into the above mentioned system we get

$$\ddot{f} + (\alpha^2 - e^2 g^2)f = 0, \quad f\dot{g} + 2\dot{f}g = 0. \tag{4.11}$$

The second ordinary differential equation from (4.11) is easily integrated

$$g = \lambda f^{-2}, \quad \lambda \in \mathbb{R}^1, \quad \lambda \neq 0. \tag{4.12}$$

Inserting the result obtained into the first ordinary differential equation from (4.11) yields the Ermakov-type equation for $f(\omega)$

$$\ddot{f} + \alpha^2 f - e^2\lambda^2 f^{-3} = 0,$$

which is integrated in elementary functions [63]

$$f = \Big(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2}\sin 2|\alpha|\omega\Big)^{1/2}. \tag{4.13}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (4.10), (4.12), (4.13) into the corresponding ansatz for $\mathbf{A}_\mu(x)$ we get the following class of exact solutions of the Yang-Mills equations (3.1):

$$\begin{aligned}\mathbf{A}_\mu &= \mathbf{e}_1 k_\mu \exp\,(-\alpha cx - \alpha w_2)\Big(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2} \\ &\quad\times \sin 2|\alpha|(bx + w_1)\Big)^{1/2} + \mathbf{e}_2\lambda\Big(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2} \\ &\quad\times \sin 2|\alpha|(bx + w_1)\Big)^{-1}\Big(b_\mu + (1/2)k_\mu \dot{w}_1\Big).\end{aligned}$$

In a similar way we have obtained the five other classes of exact solutions of the Yang-Mills equations

$$\begin{aligned}\mathbf{A}_\mu &= \mathbf{e}_1 k_\mu e^{-T}(bx\cos w_1 + cx\sin w_1 + w_2)^{1/2} Z_{1/4}\Big((ie\lambda/2)(bx\cos w_1 \\ &\quad + cx\sin w_1 + w_2)^2\Big) + \mathbf{e}_2\lambda\ (bx\cos w_1 + cx\sin w_1 + w_2) \\ &\quad\times\Big(c_\mu\cos w_1 - b_\mu\sin w_1 + 2k_\mu[(1/4)(\varepsilon e^T + \dot{T})(bx\sin w_1 \\ &\quad - cx\cos w_1) + w_3]\Big);\end{aligned}$$

$$\begin{aligned}\mathbf{A}_\mu &= \mathbf{e}_1 k_\mu e^{-T}\Big(C_1\cosh[e\lambda(bx\cos w_1 + cx\sin w_1 + w_2)] + C_2\sinh[e\lambda \\ &\quad\times(bx\cos w_1 + cx\sin w_1 + w_2)]\Big) + \mathbf{e}_2\lambda\Big(c_\mu\cos w_1 - b_\mu\sin w_1 \\ &\quad + 2k_\mu[(1/4)(\varepsilon e^T + \dot{T})(bx\sin w_1 - cx\cos w_1) + w_3]\Big);\end{aligned}$$

$$\begin{aligned}\mathbf{A}_\mu &= \mathbf{e}_1 k_\mu e^{-T}\Big(C^2(bx\cos w_1 + cx\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\Big)^{1/2} \\ &\quad + \mathbf{e}_2\lambda\Big(C^2(bx\cos w_1 + cx\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\Big)^{-1} \\ &\quad\times\Big(b_\mu\cos w_1 + c_\mu\sin w_1 - (1/2)k_\mu[\dot{w}_1(bx\sin w_1 \\ &\quad - cx\cos w_1) - \dot{w}_2]\Big);\end{aligned}$$

$$\begin{aligned}\mathbf{A}_\mu &= \mathbf{e}_1 k_\mu Z_0\Big((ie\lambda/2)[(bx + w_1)^2 + (cx + w_2)^2]\Big) + \mathbf{e}_2\lambda\Big(c_\mu(bx + w_1) \\ &\quad - b_\mu(cx + w_2) - (1/2)k_\mu[\dot{w}_1(cx + w_2) - \dot{w}_2(bx + w_1)]\Big);\end{aligned}$$

$$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu\Big(C_1[(bx + w_1)^2 + (cx + w_2)^2]^{e\lambda/2} + C_2[(bx + w_1)^2$$

$$+(cx+w_2)^2]^{-e\lambda/2}\Big) + \mathbf{e}_2\lambda[(bx+w_1)^2+(cx+w_2)^2]^{-1}$$
$$\times\Big(c_\mu(bx+w_1) - b_\mu(cx+w_2) - (1/2)k_\mu[\dot{w}_1(cx+w_2)$$
$$-\dot{w}_2(bx+w_1)]\Big).$$

Here $C_1,\ C_2,\ C\neq 0, \lambda$ are arbitrary parameters; $w_1,\ w_2,\ w_3$ are arbitrary smooth functions on $\xi=(1/2)kx$; $T=T(\xi)$ is a solution of ordinary differential equation (4.8). Besides that, we use the following notations:

$$\begin{aligned}&kx = k_\mu x^\mu, \quad bx = b_\mu x^\mu, \quad cx = c_\mu x^\mu,\\ &Z_s(\omega) = C_1 J_s(\omega) + C_2 Y_s(\omega),\\ &\mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0),\end{aligned}$$

where $J_s,\ Y_s$ are the Bessel functions.

Thus, we have obtained the broad families of exact non-Abelian solutions of the Yang-Mills equations (3.1). It can be verified by direct and rather involved computation that the solutions obtained are not self-dual, i.e., that they do not satisfy the self-dual Yang-Mills equations.

4.3 Conditional symmetry formalism

Now we briefly discuss the problem of conditional symmetry interpretation of ansatzes (3.8), (3.9), (4.7). Consider, as an example, the ansatz determined by the formulae 1 from (4.7). As the direct computation shows, generators of a three-parameter Lie group G leaving it invariant are of the form

$$\begin{aligned}&Q_1 = k_\alpha \partial_\alpha,\\ &Q_2 = b_\alpha\partial_\alpha - 2[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}},\\ &Q_3 = c_\alpha\partial_\alpha - 2[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}}.\end{aligned} \tag{4.14}$$

Evidently, system of partial differential equations (3.1) is invariant under the one-parameter group G_1 having the generator Q_1. However, it is not invariant under the one-parameter groups G_2, G_3 having the generators $Q_2,\ Q_3$. Consider, as an example, the generator Q_2. Acting by the second prolongation of the operator Q_2 (which is constructed in the standard way, see, for example, [19, 20]) on the system of partial differential equations (3.1) we see that the resulting expression does not vanish on the solution set of the equations (3.1). However, if we consider the constrained Yang-Mills equations

$$\mathbf{L}_\mu = \mathbf{0}, \quad Q_a\mathbf{A}_\mu = \mathbf{0}, \quad a=1,2,3,$$

then we see that the system obtained is invariant under the group G_2. In the above formulae we use the designations

$$\begin{aligned}
\mathbf{L}_\mu &\equiv \Box\mathbf{A}_\mu - \partial^\mu\partial_\nu\mathbf{A}_\nu + e\Big((\partial_\nu\mathbf{A}_\nu)\times\mathbf{A}_\mu - 2(\partial_\nu\mathbf{A}_\mu)\times\mathbf{A}_\nu \\
&\quad +(\partial^\mu\mathbf{A}_\nu)\times\mathbf{A}^\nu\Big) + e^2\mathbf{A}_\nu\times(\mathbf{A}^\nu\times\mathbf{A}_\mu), \\
Q_1\mathbf{A}_\mu &\equiv k_\alpha\partial_\alpha\mathbf{A}_\mu, \\
Q_2\mathbf{A}_\mu &\equiv b_\alpha\partial_\alpha\mathbf{A}_\mu + 2\Big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\Big)\mathbf{A}^\nu, \\
Q_3\mathbf{A}_\mu &\equiv c_\alpha\partial_\alpha\mathbf{A}_\mu + 2\Big(w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\Big)\mathbf{A}^\nu.
\end{aligned}$$

The same assertion holds for the Lie transformation group G_3 having the generator Q_3. Consequently, the Yang-Mills equations are conditionally-invariant with respect to the three-parameter Lie transformation group $G = G_1\otimes G_2\otimes G_3$. It means that solutions of the Yang-Mills equations obtained with the help of the ansatz invariant under the group with generators (4.14) cannot be found by means of the classical symmetry reduction procedure. We refer the reader interested in further details to the monographs [21, 33].

As very cumbersome computations show, the ansatzes determined by the formulae 2–4 from (4.7) also correspond to conditional symmetry of the Yang-Mills equations. Hence it follows, in particular, that the Yang-Mills equations should be included into the long list of mathematical and theoretical physics equations possessing non-trivial conditional symmetry [21].

5 Symmetry reduction and exact solutions of the Maxwell equations

In this part of the paper we exploit symmetry properties of the (vacuum) Maxwell equations in order to construct their exact solutions.

It is well-known that the electro-magnetic field for the case of the vanishing current is described by the Maxwell equations in vacuum

$$\begin{aligned}
\operatorname{rot}\mathbf{E} &= -\frac{\partial\mathbf{H}}{\partial x_0}, \quad \operatorname{div}\mathbf{H} = 0, \\
\operatorname{rot}\mathbf{H} &= \frac{\partial\mathbf{E}}{\partial x_0}, \quad \operatorname{div}\mathbf{E} = 0
\end{aligned} \tag{5.1}$$

for the vector fields $\mathbf{E} = \mathbf{E}(x_0,\mathbf{x})$ and $\mathbf{H} = \mathbf{H}(x_0,\mathbf{x})$ (in the sequel, we call them the Maxwell fields).

First, we give a brief overview of symmetry properties of equations (5.1) following [48].

5.1 Symmetry of the Maxwell equations

As we have mentioned in the introduction, the maximal symmetry group admitted by equations (5.1) is the sixteen-parameter group $C(1,3)\bigotimes H$. This group is the direct product of the conformal group $C(1,3)$ generated by the Lie vector fields

$$
\begin{aligned}
&P_\mu = \partial_{x_\mu}, \quad J_{0a} = x_0\partial_{x_a} + x_a\partial_{x_0} + \varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c}),\\
&J_{ab} = x_b\partial_{x_a} - x_a\partial_{x_b} + E_b\partial_{E_a} - E_a\partial_{E_b} + H_b\partial_{H_a} - H_a\partial_{H_b},\\
&D = x_\mu\partial_{x_\mu} - 2(E_a\partial_{E_a} + H_a\partial_{H_a}),\\
&K_0 = 2x_0 D - x_\mu x^\mu\partial_{x_0} + 2x_a\varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c}),\\
&K_a = -2x_a D - x_\mu x^\mu\partial_{x_a} - 2x_0\varepsilon_{abc}(E_b\partial_{H_c} - H_b\partial_{E_c})\\
&-2H_a(x_b\partial_{H_b}) - 2E_a(x_b\partial_{E_b}) + 2(x_bH_b)\partial_{H_a} + 2(x_bE_b)\partial_{E_a},
\end{aligned}
\tag{5.2}
$$

and of the one-parameter Heviside-Larmor-Rainich group H having the generator

$$
Q = E_a\partial_{H_a} - H_a\partial_{E_a}, \tag{5.3}
$$

where ε_{abc} is the third-order anti-symmetric tensor with $\varepsilon_{123} = 1$. In this section the indices denoted by the Latin alphabet letters a, b, c take the values $1, 2, 3$, and the ones denoted by the Greek alphabet letters take the values $0, 1, 2, 3$, and the summation convention is used.

It is readily seen from (5.2), (5.3) that the action of the group $C(1,3)\bigotimes H$ in the space $R^{1,3}\times R^6$, where $R^{1,3}$ is Minkowski space of the variables x_0, $\mathbf{x} = (x_1, x_2, x_3)$ and R^6 is the six-dimensional space of the functions $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{H} = (H_1, H_2, H_3)$, is projective. And furthermore, the basis generators of this group can be represented in the form (2.11).

The matrices $S_{\mu\nu}$ read as

$$
\begin{aligned}
&S_{01} = \begin{pmatrix} 0 & \tilde S_{23} \\ -\tilde S_{23} & 0 \end{pmatrix}, \qquad S_{02} = \begin{pmatrix} 0 & -\tilde S_{13} \\ \tilde S_{13} & 0 \end{pmatrix},\\
&S_{03} = \begin{pmatrix} 0 & \tilde S_{12} \\ -\tilde S_{12} & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} \tilde S_{12} & 0 \\ 0 & \tilde S_{12} \end{pmatrix},\\
&S_{13} = \begin{pmatrix} \tilde S_{13} & 0 \\ 0 & \tilde S_{13} \end{pmatrix}, \qquad S_{23} = \begin{pmatrix} \tilde S_{23} & 0 \\ 0 & \tilde S_{23} \end{pmatrix},
\end{aligned}
\tag{5.4}
$$

where 0 is the zero 3×3 matrix and

$$
\tilde S_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde S_{13} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde S_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
$$

E is the unit 6×6 matrix. The matrix $-A$ corresponding to operator Q (5.3) is given by the formula

$$
A = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{5.5}
$$

where 0 and I are zero and unit 3×3 matrices, correspondingly.

Hence, it follows that $C(1,3) \bigotimes H$-invariant ansatzes for the Maxwell fields, that reduce (5.1) to systems of ordinary differential equations, can be represented in the form (2.18), namely,

$$\mathbf{V} = \Lambda(x_0, \mathbf{x})\tilde{\mathbf{V}}(\omega) \tag{5.6}$$

with

$$\mathbf{V} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad \tilde{\mathbf{V}} = \begin{pmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \\ \tilde{H}_1 \\ \tilde{H}_2 \\ \tilde{H}_3 \end{pmatrix}.$$

Here $\Lambda(x_0, \mathbf{x})$ is the 6×6 matrix, which is non-singular in some open domain of the space $R^{1,3}$ and $\tilde{E}_a = \tilde{E}_a(\omega)$, $\tilde{H}_a = \tilde{H}_a(\omega)$ are new unknown functions of the variable $\omega = \omega(x_0, \mathbf{x})$.

In addition, the Maxwell equations admit the following discrete symmetry group [48]:

$$\Psi : \overline{x}_\mu = -x_\mu, \quad \overline{\mathbf{E}} = -\mathbf{E}, \quad \overline{\mathbf{H}} = -\mathbf{H}. \tag{5.7}$$

The transformation properties of operators (5.2), (5.3) with respect to the action of the group Ψ read as:

$$P_\mu \to -P_\mu, \quad J_{\mu\nu} \to J_{\mu\nu}, \quad D \to D, \quad K_\mu \to -K_\mu, \quad Q \to Q.$$

So that actions of discrete symmetry groups Ψ (5.7) and Φ_1 from Table 2.1 on the basis operators of the algebra $\tilde{p}(1,3)$ coincide. Therefore, we can use Assertions 2.5, 2.6 and choose the parameter j to be equal to 2, i.e., $(-1)^j = 1$.

In what follows we exploit invariance of the Maxwell equations under the conformal group $C(1,3)$ in order to construct their invariant solutions.

5.2 Conformally-invariant ansatzes for the Maxwell fields

First we will give two assertions, that simplify substantially the problem of the full description of invariant solutions of the Maxwell equations.

Assertion 5.1 *If* $\mathbf{E} = \mathbf{E}(x_0, x_3)$, $\mathbf{H} = \mathbf{H}(x_0, x_3)$, *then it is possible to construct the general solution of equations (5.1). Is has the form*

$$\begin{aligned} E_1 &= \varphi_1(\xi) + \psi_1(\eta), \quad H_1 = -\varphi_2(\xi) + \psi_2(\eta), \\ E_2 &= \varphi_2(\xi) + \psi_2(\eta), \quad H_2 = \varphi_1(\xi) - \psi_1(\eta), \\ E_3 &= C_1, \quad H_3 = C_2, \end{aligned}$$

where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are arbitrary smooth functions; $\xi = x_0 - x_3$, $\eta = x_0 + x_3$; $C_1, C_2 \in \mathbf{R}$.

Assertion 5.2 *If $\mathbf{E} = \mathbf{E}(x_1, x_2, \xi)$, $\mathbf{H} = \mathbf{H}(x_1, x_2, \xi)$, where $\xi = \frac{1}{2}(x_0 - x_3)$, then it is possible to construct the general solution of the Maxwell equations (5.1). It is given by the following formulae:*

$$\begin{aligned} E_1 &= \frac{1}{2}(R + R^* + T_1 + T_1^*), \quad H_1 = \frac{1}{2}(iR - iR^* - T_2 - T_2^*), \\ E_2 &= \frac{1}{2}(iR - iR^* + T_2 + T_2^*), \quad H_2 = \frac{1}{2}(R + R^* - T_1 - T_1^*), \\ E_3 &= S + S^*, \quad H_3 = iS - iS^*, \end{aligned}$$

where

$$T_j = \frac{\partial^2 \theta_j}{\partial \xi^2}, \quad (j = 1, 2), \quad S = \frac{\partial \theta_1}{\partial \xi} + i\frac{\partial \theta_2}{\partial \xi} + \lambda(z),$$
$$R = -2\Big(\frac{\partial \theta_1}{\partial z} + i\frac{\partial \theta_2}{\partial z}\Big) + \xi\frac{d\lambda}{dz}.$$

Here $\theta_j = \theta_j(z, \xi)$, $\lambda(z)$ are arbitrary functions analytic by the variable $z = x_1 + ix_2$; $j = 1, 2$; i is the imaginary unit, i.e., $i^2 = -1$.

Proof of the above assertions can be found in [50]–[53].

It follows from Assertions 5.1, 5.2 that we have to exclude from the further considerations those subalgebras of the conformal algebra that yield solutions of the form covered by these assertions. It is straightforward to check that we have to skip subalgebras L of the rank 3 fulfilling the conditions

$$< P_0 + P_3 > \not\subset L, \quad < P_0 - P_3 > \not\subset L, \quad < P_0, P_3 > \not\subset L, \quad < P_1, P_2 > \not\subset L.$$

Owing to this fact, to get the full description of conformally-invariant solutions of the Maxwell equations it suffices to consider the following subalgebras of the conformal algebra $c(1,3)$ (note, that we have also made use of the discrete symmetry group Ψ in order to simplify their basis elements):

$$\begin{aligned} &M_1 = \langle J_{03}, G_1, P_2 \rangle; \quad M_2 = \langle G_1, G_2, J_{03} + \alpha J_{12} \rangle, \quad \alpha \in \mathbf{R}; \\ &M_3 = \langle J_{12}, D, P_0 \rangle; \quad M_4 = \langle J_{12}, D, P_3 \rangle, \\ &M_5 = \langle J_{03}, D, P_1 \rangle; \quad M_6 = \langle J_{03}, J_{12}, D \rangle; \\ &M_7 = \langle G_1, J_{03} + \alpha D, P_2 \rangle \quad (0 < |\alpha| \le 1); \\ &M_8 = \langle J_{03} - D + M, G_1, P_2 \rangle; \quad M_9 = \langle J_{03} + 2D, G_1 + 2T, P_2 \rangle; \\ &M_{10} = \langle J_{12}, S + T, Z \rangle; \quad M_{11} = \langle S + T + J_{12}, Z, G_1 + P_2 \rangle; \\ &M_{12} = \langle P_2 + K_2 + \sqrt{3}(P_1 + K_1) + K_0 - P_0, J_{02} - D - \sqrt{3}J_{01}, \\ &\quad P_0 + K_0 - 2(K_2 - P_2) \rangle; \\ &M_{13} = \langle P_0 + K_0 \rangle \bigoplus \langle J_{12}, K_3 - P_3 \rangle; \\ &M_{14} = \langle 2J_{12} + K_3 - P_3, 2J_{13} - K_2 + P_2, 2J_{23} + K_1 - P_1 \rangle; \\ &M_{15} = \langle P_1 + K_1 + 2J_{03}, P_2 + K_2 + K_0 - P_0, 2J_{12} + K_3 - P_3 \rangle. \end{aligned}$$

Here we use the following designations:

$$M = P_0 + P_3, \quad G_{0j} = J_{0j} - J_{j3}, \ (j = 1, 2),$$
$$Z = J_{03} + D, \quad S = \frac{1}{2}(K_0 + K_3), \quad T = \frac{1}{2}(P_0 - P_3).$$

In the sequel, we consider the first ten subalgebras from the above list. For these subalgebras we can represent the matrix Λ from ansatz (5.6) as follows

$$\Lambda = \exp\{(\ln\theta)E\}\exp(2\theta_1 H_1)\exp(2\theta_2 H_2)\exp(-\theta_0 S_{03})\exp(\theta_3 S_{12}),$$

the matrices $S_{\mu\nu}$ having the form (5.4). So that, we have

$$\Lambda = \theta\begin{pmatrix} C & G \\ -G & C \end{pmatrix},$$

where

$$\begin{aligned} C &= \begin{pmatrix} \cosh\theta_0\cos\theta_3 - r_1 & -\cosh\theta_0\sin\theta_3 + r_2 & 2\theta_1 \\ \cosh\theta_0\sin\theta_3 + r_2 & \cosh\theta_0\cos\theta_3 + r_1 & 2\theta_2 \\ -2s_1 & 2s_2 & 1 \end{pmatrix}, \\ G &= \begin{pmatrix} \sinh\theta_0\sin\theta_3 + r_2 & \sinh\theta_0\cos\theta_3 + r_1 & 2\theta_2 \\ -\sinh\theta_0\cos\theta_3 + r_1 & \sinh\theta_0\sin\theta_3 - r_2 & -2\theta_1 \\ 2s_2 & 2s_1 & 0 \end{pmatrix}, \end{aligned}$$

and furthermore,

$$\begin{aligned} r_1 &= 2[(\theta_1^2 - \theta_2^2)\cos\theta_3 + 2\theta_1\theta_2\sin\theta_3]e^{-\theta_0}, \\ r_2 &= 2[(\theta_1^2 - \theta_2^2)\sin\theta_3 - 2\theta_1\theta_2\cos\theta_3]e^{-\theta_0}, \\ s_1 &= 2[\theta_1\cos\theta_3 + \theta_2\sin\theta_3]e^{-\theta_0}, \\ s_2 &= 2[\theta_1\sin\theta_3 - \theta_2\cos\theta_3]e^{-\theta_0}. \end{aligned}$$

After some algebra we obtain the following form of the conformally-invariant ansatz for the Maxwell fields:

$$\begin{aligned} E_1 &= \theta\{(\tilde{E}_1\cos\theta_3 - \tilde{E}_2\sin\theta_3)\cosh\theta_0 \\ &\quad +(\tilde{H}_1\sin\theta_3 + \tilde{H}_2\cos\theta_3)\sinh\theta_0 \\ &\quad +2\theta_1\tilde{E}_3 + 2\theta_2\tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\}, \\ E_2 &= \theta\{(\tilde{E}_2\cos\theta_3 + \tilde{E}_1\sin\theta_3)\cosh\theta_0 \\ &\quad +(\tilde{H}_2\sin\theta_3 - \tilde{H}_1\cos\theta_3)\sinh\theta_0 \\ &\quad -2\theta_1\tilde{H}_3 + 2\theta_2\tilde{E}_3 + 4\theta_1\theta_2\Sigma_2 - 2(\theta_1^2 - \theta_2^2)\Sigma_1\}, \\ E_3 &= \theta\{\tilde{E}_3 + 2\theta_1\Sigma_2 + 2\theta_2\Sigma_1\}, \\ H_1 &= \theta\{(\tilde{H}_1\cos\theta_3 - \tilde{H}_2\sin\theta_3)\cosh\theta_0 \end{aligned} \tag{5.8}$$

$$
\begin{aligned}
& -(\tilde{E}_1 \sin\theta_3 + \tilde{E}_2 \cos\theta_3) \sinh\ \theta_0 \\
& +2\theta_1 \tilde{H}_3 - 2\theta_2 \tilde{E}_3 - 4\theta_1\theta_2\Sigma_2 + 2(\theta_1^2 - \theta_2^2)\Sigma_1\}, \\
H_2 \ = \ & \theta\{(\tilde{H}_2 \cos\theta_3 + \tilde{H}_1 \sin\theta_3)\cosh\theta_0 \\
& +(\tilde{E}_1 \cos\theta_3 - \tilde{E}_2 \sin\theta_3)\sinh\theta_0 \\
& +2\theta_1 \tilde{E}_3 + 2\theta_2 \tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\}, \\
H_3 \ = \ & \theta\{\tilde{H}_3 + 2\theta_1\Sigma_1 - 2\theta_2\Sigma_2\}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\Sigma_1 &= [(\tilde{H}_2 - \tilde{E}_1)\sin\theta_3 - (\tilde{E}_2 + \tilde{H}_1)\cos\theta_3]e^{-\theta_0}, \\
\Sigma_2 &= [(\tilde{E}_2 + \tilde{H}_1)\sin\theta_3 + (\tilde{H}_2 - \tilde{E}_1)\cos\theta_3]e^{-\theta_0}.
\end{aligned}
$$

The form of the functions $\theta,\ \theta_\mu, \omega$ for each of the subalgebras $M_j,\ (j = 1, 2, \ldots, 10)$ is obtained from Assertions 2.5–2.7 with $k = 2$.

$$
\begin{aligned}
M_1 \ : \ & \theta = 1,\ \theta_0 = -\ln|x_0 - x_3|,\ \theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}, \\
& \theta_2 = \theta_3 = 0,\ \omega = x_0^2 - x_1^2 - x_3^2; \\
M_2 \ : \ & \theta = 1,\ \theta_0 = -\ln|x_0 - x_3|,\ \theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}, \\
& \theta_2 = -\frac{1}{2}x_2(x_0 - x_3)^{-1}, \theta_3 = \alpha\ln|x_0 - x_3|, \\
& \omega = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad \alpha \in \mathbf{R}; \\
M_3 \ : \ & \theta = (x_3)^{-2},\ \theta_0 = \theta_1 = \theta_2 = 0, \\
& \theta_3 = \arctan\frac{x_2}{x_1},\ \omega = (x_1^2 + x_2^2)x_3^{-2}; \\
M_4 \ : \ & \theta = (x_0)^{-2},\ \theta_0 = \theta_1 = \theta_2 = 0, \\
& \theta_3 = \arctan\frac{x_2}{x_1},\ \omega = (x_1^2 + x_2^2)x_0^{-2}; \\
M_5 \ : \ & \theta = (x_2)^{-2},\ \theta_0 = \ln|(x_0 + x_3)x_2^{-1}|,\ \theta_1 = \theta_2 = \theta_3 = 0, \\
& \omega = (x_0^2 - x_3^2)x_2^{-2}; \\
M_6 \ : \ & (x_1^2 + x_2^2)^{-1},\ \theta_0 = -\frac{1}{2}\ln|(x_0 - x_3)(x_0 + x_3)^{-1}|, \\
& \theta_1 = \theta_2 = 0,\ \theta_3 = \arctan\frac{x_2}{x_1},\ \omega = (x_1^2 + x_2^2)(x_0^2 - x_3^2)^{-1}; \\
M_7 \ : \ & 1)\ \alpha = -1 \\
& \theta = (x_0 - x_3)^{-1},\ \theta_0 = -\frac{1}{2}\ln|x_0 - x_3|, \\
& \theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1},\ \theta_2 = \theta_3 = 0, \\
& \omega = x_0 + x_3 - x_1^2(x_0 - x_3)^{-1}; \\
& 2)\ \alpha \neq -1
\end{aligned}
$$

$$
\begin{aligned}
& \theta=|x_0^2-x_1^2-x_3^2|^{-1},\ \theta_0=\frac{1}{2\alpha}\ln|x_0^2-x_1^2-x_3^2|,\\
& \theta_1=-\frac{1}{2}x_1(x_0-x_3)^{-1},\ \theta_2=\theta_3=0,\\
& \omega=2\alpha\ln|x_0-x_3|+(1-\alpha)\ln|x_0^2-x_1^2-x_3^2|;\\
M_8\ :\ & \theta=|x_0-x_3|^{-1},\ \theta_0=-\frac{1}{2}\ln|x_0-x_3|,\ \theta_1=-\frac{1}{2}x_1(x_0-x_3)^{-1},\\
& \theta_2=\theta_3=0,\ \omega=x_0+x_3-x_1^2(x_0-x_3)^{-1}+\ln|x_0-x_3|;\\
M_9\ :\ & \theta=[(x_0-x_3)^2-4x_1]^{-2},\ \theta_0=\frac{1}{2}\ln|(x_0-x_3)^2-4x_1|,\\
& \theta_1=-\frac{1}{4}(x_0-x_3),\ \theta_2=\theta_3=0,\\
& \omega=[x_0+x_3-x_1(x_0-x_3)+\frac{1}{6}(x_0-x_3)^3]^2[(x_0-x_3)^2-4x_1]^{-3};\\
M_{10}\ :\ & \theta=[(x_1-(x_0-x_3)x_2)^2(1+(x_0-x_3)^2)^{-1}]^{-1};\\
& \theta_0=\frac{1}{2}\ln[(x_1-(x_0-x_3)x_2)^2(1+(x_0-x_3)^2)^{-3}],\\
& \theta_1=-\frac{1}{2}(x_2+(x_0-x_3)x_1)(1+(x_0-x_3)^2)^{-1},\\
& \theta_2=\frac{1}{2}(x_1-(x_0-x_3)x_2)(1+(x_0-x_3)^2)^{-1},\\
& \theta_3=-\arctan(x_0-x_3),\ \omega=[(x_0+x_3)(1+(x_0-x_3)^2)^2\\
& \quad -2x_1(x_2+(x_0-x_3)x_1)-(x_0-x_3)(x_1^2(x_0-x_3)^2-x_2^2)]\\
& \quad \times[x_1-(x_0-x_3)x_2]^{-2}-x_0+x_3.
\end{aligned}
$$

5.3 Exact solutions of the Maxwell equations

Now we have to insert ansatzes (5.8) into (5.1). However, it is more convenient to rewrite the Maxwell equations (5.1) in the following equivalent form:

$$
\begin{aligned}
& \partial_{x_1}(E_1+H_2)+\partial_{x_2}(E_2-H_1)=(\partial_{x_0}-\partial_{x_3})E_3,\\
& \partial_{x_1}(E_1-H_2)+\partial_{x_2}(E_2+H_1)=-(\partial_{x_0}+\partial_{x_3})E_3,\\
& \partial_{x_1}(E_2-H_1)-\partial_{x_2}(E_1+H_2)=-(\partial_{x_0}-\partial_{x_3})H_3,\\
& \partial_{x_1}(E_2+H_1)-\partial_{x_2}(E_1-H_2)=-(\partial_{x_0}+\partial_{x_3})H_3,\\
& (\partial_{x_0}+\partial_{x_3})(E_1+H_2)=\partial_{x_1}E_3+\partial_{x_2}H_3,\\
& (\partial_{x_0}-\partial_{x_3})(E_1-H_2)=-\partial_{x_1}E_3+\partial_{x_2}H_3,\\
& (\partial_{x_0}-\partial_{x_3})(E_2+H_1)=-\partial_{x_2}E_3-\partial_{x_1}H_3,\\
& (\partial_{x_0}+\partial_{x_3})(E_2-H_1)=\partial_{x_2}E_3-\partial_{x_1}H_3.
\end{aligned}
\tag{5.9}
$$

We will give the calculation details for the case of the subalgebra M_1 only, since the remaining subalgebras are handled in the similar way. For the case

in hand, ansatz (5.8) can be written in the form

$$\begin{aligned}
&E_1 + H_2 = f e^{\theta_0} + 4\theta_1 \tilde{E}_3 - 4\theta_1^2 e^{-\theta_0} h, \\
&E_1 - H_2 = h e^{-\theta_0}, \quad E_2 + H_1 = \rho e^{-\theta_0}, \\
&E_2 - H_1 = g e^{\theta_0} - 4\theta_1 \tilde{H}_3 + 4\theta_1^2 e^{-\theta_0} \rho, \\
&E_3 = \tilde{E}_3 - 2\theta_1 h e^{-\theta_0}, \quad H_3 = \tilde{H}_3 - 2\theta_1 \rho e^{-\theta_0},
\end{aligned} \tag{5.10}$$

where $\theta_0 = -\ln|x_0 - x_3|$, $\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}$. The functions $\tilde{E}_3$, $\tilde{H}_3$ and

$$\begin{aligned}
f &= f(\omega) = \tilde{E}_1 + \tilde{H}_2, \quad g = g(\omega) = \tilde{E}_2 - \tilde{H}_1, \\
h &= h(\omega) = \tilde{E}_1 - \tilde{H}_2, \quad \rho = \rho(\omega) = \tilde{E}_2 + \tilde{H}_1
\end{aligned} \tag{5.11}$$

are arbitrary smooth functions of the variable $\omega = x_0^2 - x_1^2 - x_3^2$.

Inserting (5.10) into the second and fourth equations from (5.9) gives equations

$$\dot{\tilde{E}}_3 = 0, \quad \dot{\tilde{H}}_3 = 0. \tag{5.12}$$

We remind that the dot over symbol stands for the derivative with respect to the variable ω.

Similarly, we get from the sixth and seventh equations of system (5.9) the following reduced equations:

$$2\omega \dot{h} + 3h = 0, \quad 2\omega\dot{\rho} + 3\rho = 0. \tag{5.13}$$

Next, the fifth and eighth equations give rise to ordinary differential equations of the form

$$2\dot{f} - h = 0, \quad 2\dot{g} + \rho = 0. \tag{5.14}$$

Finally, substituting ansatz (5.10) into the first and third equations from (5.9) yields

$$\begin{aligned}
&4\varepsilon\theta_1[\omega\dot{h} + h + \dot{f}] = 2\xi^{-1}\tilde{E}_3, \\
&4\varepsilon\theta_1[\dot{g} - \omega\dot{\rho} - \rho] = -2\xi^{-1}\tilde{H}_3,
\end{aligned} \tag{5.15}$$

where $\varepsilon = 1$ for $\xi = x_0 - x_3 > 0$ and $\varepsilon = -1$ for $x_0 - x_3 < 0$.

Taking into account (5.13), (5.14) we see that

$$\tilde{E}_3 = 0, \quad \tilde{H}_3 = 0.$$

Summing up we conclude that the ansatz invariant with respect to the subalgebra M_1 reduce the Maxwell equations to the following system of ordinary differential equations:

$$\begin{aligned}
&2\omega\dot{h} + 3h = 0, \quad 2\omega\dot{\rho} + 3\rho = 0, \quad 2\dot{f} - h = 0, \\
&2\dot{g} + \rho = 0, \quad \tilde{E}_3 = 0, \quad \tilde{H}_3 = 0.
\end{aligned}$$

Below we give the reduced systems for the ansatzes invariant with respect to the remaining subalgebras M_2–M_{10}. Note that the functions f, g, h, ρ are of the form (5.11).

$$
\begin{array}{ll}
1. & \text{System (5.16).} \\
2. & \dot{f}=0,\ \tilde{E}_3=0,\ \dot{g}=0,\ \tilde{H}_3=0, \\
 & \omega\dot{h}+2h+\alpha\rho=0,\ \omega\dot{\rho}+2\rho-\alpha h=0,\ \alpha\in\mathbf{R}. \\
3. & 2\omega(1+\omega)\ddot{\tilde{E}}_3+(7\omega+2)\dot{\tilde{E}}_3+3\tilde{E}_3=0, \\
 & f=h=-2\sqrt{\omega}(\tilde{E}_3+(1+\omega)\dot{\tilde{E}}_3), \\
 & 2\omega(1+\omega)\ddot{\tilde{H}}_3+(7\omega+2)\dot{\tilde{H}}_3+3\tilde{H}_3=0, \\
 & g=-\rho=2\sqrt{\omega}(\tilde{H}_3+(1+\omega)\dot{\tilde{H}}_3). \\
4. & 2\omega(\omega-1)\ddot{\tilde{E}}_3+(7\omega-2)\dot{\tilde{E}}_3+3\tilde{E}_3=0, \\
 & f=-h=2\sqrt{\omega}(\tilde{E}_3+(\omega-1)\dot{\tilde{E}}_3), \\
 & 2\omega(\omega-1)\ddot{\tilde{H}}_3+(7\omega-2)\dot{\tilde{H}}_3+3\tilde{H}_3=0, \\
 & g=\rho=-2\sqrt{\omega}(\tilde{H}_3+(\omega-1)\dot{\tilde{H}}_3). \\
5. & 2\omega(\omega-1)\ddot{\tilde{E}}_3+(7\omega-2)\dot{\tilde{E}}_3+3\tilde{E}_3=0, \\
 & g=-\omega^{-1}\rho=2\varepsilon[\tilde{E}_3+(\omega-1)\dot{\tilde{E}}_3], \\
 & 2\omega(\omega-1)\ddot{\tilde{H}}_3+(7\omega-2)\dot{\tilde{H}}_3+3\tilde{H}_3=0, \\
 & f=\omega^{-1}h=2\varepsilon[\tilde{H}_3+(\omega-1)\dot{\tilde{H}}_3], \\
 & \varepsilon=1 \text{ for } (x_0+x_3)x_2^{-1}>0, \\
 & \varepsilon=-1 \text{ for } (x_0+x_3)x_2^{-1}<0. \\
6. & (\omega-1)\dot{\tilde{E}}_3+\tilde{E}_3=0, \\
 & 2\omega\dot{f}+f=-2\varepsilon_2\sqrt{|\omega|}\dot{\tilde{E}}_3, \\
 & 2\omega\dot{h}+h=2\varepsilon_1\sqrt{|\omega|}\dot{\tilde{E}}_3, \\
 & (\omega-1)\dot{\tilde{H}}_3+\tilde{H}_3=0, \\
 & 2\omega\dot{\rho}+\rho=2\varepsilon_1\sqrt{|\omega|}\dot{\tilde{H}}_3, \\
 & 2\omega\dot{g}+g=2\varepsilon_2\sqrt{|\omega|}\dot{\tilde{H}}_3, \\
 & \varepsilon_1=1 \text{ for } x_0+x_3>0,\ \varepsilon_1=-1 \text{ for } x_0+x_3<0. \\
 & \varepsilon_2=1 \text{ for } x_0-x_3>0,\ \varepsilon_2=-1 \text{ for } x_0-x_3<0. \\
7. & 1)\ \dot{\tilde{E}}_3=0,\quad 2\dot{f}=\varepsilon h, \\
 & \ \dot{\tilde{H}}_3=0,\quad 2\dot{g}=-\varepsilon\rho, \\
 & \ \varepsilon=1 \text{ for } x_0-x_3>0, \\
 & \ \varepsilon=-1 \text{ for } x_0-x_3<0.
\end{array}
\tag{5.16}
$$

$$2)\tilde{E}_3 = 0, \quad 2(1+\alpha)\dot{h} - (1+\frac{1}{\alpha})h = 0,$$
$$(\frac{1}{\alpha} - 2)f + 2(1-\alpha)\dot{f} = \varepsilon e^{-\frac{1}{\alpha}\omega}h,$$
$$\tilde{H}_3 = 0, \quad 2(1+\alpha)\dot{\rho} - (1+\frac{1}{\alpha})\rho = 0,$$
$$(\frac{1}{\alpha} - 2)g + 2(1-\alpha)\dot{g} = -\varepsilon e^{-\frac{1}{\alpha}\omega}\rho,$$
$$0 < |\alpha| \leq 1, \ \alpha \neq -1, \ \varepsilon = 1 \text{ for } x_0^2 - x_1^2 - x_3^2 > 0,$$
$$\varepsilon = -1 \text{ for } x_0^2 - x_1^2 - x_3^2 < 0.$$

8. $$\dot{h} = 0, \quad \dot{\tilde{E}}_3 = 0, \quad \dot{\rho} = 0, \quad \dot{\tilde{H}}_3 = 0,$$
$$2\varepsilon\dot{f} - h = 0, \quad 2\varepsilon\dot{g} + \rho = 0,$$
$$\varepsilon = 1 \text{ for } x_0 - x_3 > 0, \ \varepsilon = -1 \text{ for } x_0 - x_3 < 0.$$

9. $$h = 4\varepsilon f, \quad \rho = -4\varepsilon g,$$
$$\tilde{E}_3 = -(9\omega^2 + \frac{\varepsilon}{4})\dot{f} - 15\omega f = 0,$$
$$(36\omega^2 + \varepsilon)\ddot{f} + 180\omega\dot{f} + 140f = 0,$$
$$\tilde{H}_3 = 15\omega g + (9\omega^2 + \frac{\varepsilon}{4})\dot{g},$$
$$(36\omega^2 + \varepsilon)\ddot{g} + 180\omega\dot{g} + 140g = 0,$$
$$\varepsilon = 1 \text{ for } \sigma > 0, \ \varepsilon = -1 \text{ for } \sigma < 0,$$
$$\sigma = 4x_1 - (x_0 - x_3)^2.$$

10. $$\dot{f} = \dot{h}, \quad h = (\omega^2 + 1)\dot{\tilde{E}}_3 + \omega\tilde{E}_3,$$
$$(\omega^2 + 1)\ddot{\tilde{E}}_3 + 4\omega\dot{\tilde{E}}_3 + 2\tilde{E}_3 = 0,$$
$$\dot{g} = -\dot{\rho}, \quad \rho = (\omega^2 + 1)\dot{\tilde{H}}_3 + \omega\tilde{H}_3,$$
$$(\omega^2 + 1)\ddot{\tilde{H}}_3 + 4\omega\dot{\tilde{H}}_3 + 2\tilde{H}_3 = 0.$$

The above systems are linear and therefore are easily integrated (the integration details can be found in [50]–[53]). Below we give the final result. Namely, we present the families of exact solutions of the Maxwell equations (5.1) invariant with respect to the subalgebras M_1–M_{10}.

$$M_1 \ : \ E_1 = C_2(x_0 - x_3)^{-1} - 2x_3C_1|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}},$$
$$E_2 = C_4(x_0 - x_3)^{-1} + 2x_0C_3|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}},$$
$$E_3 = 2x_1C_1|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}},$$
$$H_1 = -C_4(x_0 - x_3)^{-1} - 2x_3C_3|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}},$$
$$H_2 = C_2(x_0 - x_3)^{-1} - 2x_0C_1|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}},$$
$$H_3 = 2x_1C_3|x_0^2 - x_1^2 - x_3^2|^{-\frac{3}{2}}.$$

$$
\begin{aligned}
M_2 \quad : \quad & E_1 = |\xi|^{-1}\{C_1\cos(\alpha\ln|\xi|) - C_2\sin(\alpha\ln|\xi|) \\
& \quad -x_1x_2[h\sin(\alpha\ln|\xi|) + \rho\cos(\alpha\ln|\xi|)] \\
& \quad +\frac{1}{2}(\xi^2 - x_1^2 + x_2^2)[h\cos(\alpha\ln|\xi|) - \rho\sin(\alpha\ln|\xi|)]\}, \\
& E_2 = |\xi|^{-1}\{C_2\cos(\alpha\ln|\xi|) + C_1\sin(\alpha\ln|\xi|) \\
& \quad +x_1x_2[\rho\sin(\alpha\ln|\xi|) - h\cos(\alpha\ln|\xi|)] \\
& \quad +\frac{1}{2}(\xi^2 + x_1^2 - x_2^2)[h\sin(\alpha\ln|\xi|) + \rho\cos(\alpha\ln|\xi|)]\}, \\
& E_3 = \varepsilon\{h[x_1\cos(\alpha\ln|\xi|) + x_2\sin(\alpha\ln|\xi|)] \\
& \quad +\rho[x_2\cos(\alpha\ln|\xi|) - x_1\sin(\alpha\ln|\xi|)], \\
& H_1 = |\xi|^{-1}\{-C_2\cos(\alpha\ln|\xi|) - C_1\sin(\alpha\ln|\xi|) \\
& \quad -x_1x_2[\rho\sin(\alpha\ln|\xi|) - h\cos(\alpha\ln|\xi|)] \\
& \quad +\frac{1}{2}(\xi^2 - x_1^2 + x_2^2)[h\sin(\alpha\ln|\xi|) + \rho\cos(\alpha\ln|\xi|)]\}, \\
& H_2 = |\xi|^{-1}\{C_1\cos(\alpha\ln|\xi|) - C_2\sin(\alpha\ln|\xi|) \\
& \quad -x_1x_2[h\sin(\alpha\ln|\xi|) + \rho\cos(\alpha\ln|\xi|)] \\
& \quad -\frac{1}{2}(\xi^2 + x_1^2 - x_2^2)[h\cos(\alpha\ln|\xi|) - \rho\sin(\alpha\ln|\xi|)]\}, \\
& H_3 = \varepsilon\{h[x_1\sin(\alpha\ln|\xi|) - x_2\cos(\alpha\ln|\xi|)] \\
& \quad +\rho[x_1\cos(\alpha\ln|\xi|) + x_2\sin(\alpha\ln|\xi|)], \\
& \text{where } \xi = x_0 - x_3,\ h = \omega^{-2}[C_4\cos(\alpha\ln|\omega|) - C_3\sin(\alpha\ln|\omega|)], \\
& \rho = \omega^{-2}[C_3\cos(\alpha\ln|\omega|) + C_4\sin(\alpha\ln|\omega|)],\ \omega = x_\mu x^\mu, \\
& \alpha \in \mathbf{R},\ \varepsilon = 1,\ \text{for } \xi > 0 \text{ and } \varepsilon = -1 \text{ for } \xi < 0.
\end{aligned}
$$

$$
\begin{aligned}
M_3 \quad : \quad & E_a = -\frac{2C_1x_a}{x_3(x_1^2 + x_2^2)} + x_a\sigma^{-\frac{3}{2}}A_{12}, \quad E_3 = x_3\sigma^{-\frac{3}{2}}A_{12}, \\
& H_a = -\frac{2C_3x_a}{x_3(x_1^2 + x_2^2)} + x_a\sigma^{-\frac{3}{2}}A_{34}, \quad H_3 = x_3\sigma^{-\frac{3}{2}}A_{34}, \\
& \text{where } A_{ij} = C_i(\ln\left|\frac{\sqrt{\sigma} - x_3}{\sqrt{\sigma} + x_3}\right| + 2x_3^{-1}\sqrt{\sigma}) + C_j, \\
& \sigma = x_1^2 + x_2^2 + x_3^2,\ a = 1, 2.
\end{aligned}
$$

$$
\begin{aligned}
M_4 \quad : \quad & 1)\ E_a = \varepsilon_{ab}x_b\Big\{\frac{2C_4}{x_0(x_1^2 + x_2^2)} - \sigma^{-\frac{3}{2}}A_{34}\Big\}, \quad E_3 = x_0\sigma^{-\frac{3}{2}}A_{12}; \\
& H_a = -\varepsilon_{ab}x_b\Big\{\frac{2C_2}{x_0(x_1^2 + x_2^2)} - \sigma^{-\frac{3}{2}}A_{12}\Big\}, \quad H_3 = x_0\sigma^{-\frac{3}{2}}A_{34}, \\
& \text{where } A_{ij} = C_i + C_j\Big(\ln\left|\frac{\sqrt{\sigma} - x_0}{\sqrt{\sigma} + x_0}\right| + 2x_0^{-1}\sqrt{\sigma}\Big), \\
& \sigma = x_0^2 - x_1^2 - x_2^2 > 0,\ a, b = 1, 2;
\end{aligned}
$$

$$2)\ E_a = -\varepsilon_{ab} x_b \Big\{ \frac{C_4}{x_0(x_1^2+x_2^2)} - \sigma^{-\frac{3}{2}} B_{34} \Big\}, \quad E_3 = x_0 \sigma^{-\frac{3}{2}} B_{12};$$

$$H_a = -\varepsilon_{ab} x_b \Big\{ \frac{C_2}{x_0(x_1^2+x_2^2)} - \sigma^{-\frac{3}{2}} B_{12} \Big\}, \quad H_3 = x_0 \sigma^{-\frac{3}{2}} B_{34},$$

where $B_{ij} = C_i + C_j (x_0^{-1}\sqrt{\sigma} - \arctan \frac{\sqrt{\sigma}}{x_0}),\ \sigma = x_1^2 + x_2^2 - x_0^2 > 0,$
$a, b = 1, 2.$

Here $\varepsilon_{ab},\ (a, b = 1, 2)$ is the anti − symmetric tensor of the second order with $\varepsilon_{12} = 1$.

M_5 : $1)\ E_1 = \frac{2x_0 C_4}{x_2(x_0^2 - x_3^2)} - x_0 \sigma^{-\frac{3}{2}} A_{34},\ E_2 = \frac{2x_3 C_2}{x_2(x_0^2 - x_3^2)} - x_3 \sigma^{-\frac{3}{2}} A_{12},$

$$H_1 = -\frac{2x_0 C_2}{x_2(x_0^2 - x_3^2)} + x_0 \sigma^{-\frac{3}{2}} A_{12},\ H_2 = \frac{2x_3 C_4}{x_2(x_0^2 - x_3^2)} - x_3 \sigma^{-\frac{3}{2}} A_{34},$$

$$E_3 = x_2 \sigma^{-\frac{3}{2}} A_{12}, \quad H_3 = x_2 \sigma^{-\frac{3}{2}} A_{34},$$

where $A_{ij} = C_i + C_j \Big(2\frac{\sqrt{\sigma}}{x_2} - \ln \Big| \frac{\sqrt{\sigma} - x_2}{\sqrt{\sigma} + x_2} \Big| \Big),\ \sigma = x_2^2 + x_3^2 - x_0^2 > 0;$

$$2)\ E_1 = \frac{x_0 C_4}{x_2(x_0^2 - x_3^2)} - x_0 \sigma^{-\frac{3}{2}} B_{34},\ E_2 = \frac{x_3 C_2}{x_2(x_0^2 - x_3^2)} - x_3 \sigma^{-\frac{3}{2}} B_{12},$$

$$H_1 = -\frac{x_0 C_2}{x_2(x_0^2 - x_3^2)} + x_0 \sigma^{-\frac{3}{2}} B_{12},\ H_2 = \frac{x_3 C_4}{x_2(x_0^2 - x_3^2)} - x_3 \sigma^{-\frac{3}{2}} B_{34},$$

$$E_3 = x_2 \sigma^{-\frac{3}{2}} B_{12}, \quad H_3 = x_2 \sigma^{-\frac{3}{2}} B_{34},$$

where $B_{ij} = C_i + C_j \Big(\frac{\sqrt{\sigma}}{x_2} - \arctan \frac{\sqrt{\sigma}}{x_2} \Big), \quad \sigma = x_0^2 - x_2^2 - x_3^2 > 0.$

M_6 : $E_1 = \frac{1}{2} \Big[\frac{\xi(x_1 C_2 - x_2 C_5) + \eta(x_1 C_3 - x_3 C_6)}{\xi\eta(x_1^2 + x_2^2)}$

$$- \frac{\varepsilon_1 \xi(x_1 C_1 + x_2 C_4) - \varepsilon_2 \eta(x_1 C_1 - x_2 C_4)}{\sigma(x_1^2 + x_2^2)} \Big],$$

$$E_2 = \frac{1}{2} \Big[\frac{\xi(x_1 C_5 + x_2 C_2) + \eta(x_1 C_6 + x_2 C_3)}{\xi\eta(x_1^2 + x_2^2)}$$

$$+ \frac{\varepsilon_1 \xi(x_1 C_4 - x_2 C_1) + \varepsilon_2 \eta(x_1 C_4 + x_2 C_1)}{\sigma(x_1^2 + x_2^2)} \Big],$$

$$H_1 = \frac{1}{2} \Big[\frac{\eta(x_1 C_6 + x_2 C_3) - \xi(x_1 C_5 + x_2 C_2)}{\xi\eta(x_1^2 + x_2^2)}$$

$$+ \frac{\varepsilon_1 \xi(x_2 C_1 - x_1 C_4) + \varepsilon_2 \eta(x_1 C_4 + x_2 C_1)}{\sigma(x_1^2 + x_2^2)} \Big],$$

$$H_2 = \frac{1}{2} \Big[\frac{\xi(x_1 C_2 - x_2 C_5) - \eta(x_1 C_3 - x_2 C_6)}{\xi\eta(x_1^2 + x_2^2)}$$

$$-\frac{\varepsilon_1\xi(x_1C_1+x_2C_4)+\varepsilon_2\eta(x_1C_1-x_2C_4)}{\sigma(x_1^2+x_2^2)}\Big],$$

$E_3=C_1\sigma^{-1},\quad H_3=C_4\sigma^{-1},$

where $\sigma=x_1^2+x_2^2+x_3^2-x_0^2,\quad \xi=x_0+x_3, \eta=x_0-x_3,$

$$\varepsilon_1=\begin{cases}1, \text{ if } x_0+x_3>0,\\ -1, \text{ if } x_0+x_3<0,\end{cases}\quad \varepsilon_2=\begin{cases}1, \text{ if } x_0-x_3>0,\\ -1, \text{ if } x_0-x_3<0.\end{cases}$$

M_7 : 1) $\alpha=-1$

$$E_1=|\eta|^{-\frac{3}{2}}(C_1+\frac{1}{4}F)-x_1\eta^{-2}C_2-\frac{1}{2}\varepsilon|\eta|^{-\frac{1}{2}}f(x_1^2\eta^{-2}-1),$$

$$E_2=|\eta|^{-\frac{3}{2}}(C_3-\frac{1}{4}G)+x_1\eta^{-2}C_4+\frac{1}{2}\varepsilon|\eta|^{-\frac{1}{2}}g(x_1^2\eta^{-2}+1),$$

$$H_1=-|\eta|^{-\frac{3}{2}}(C_3-\frac{1}{4}G)-x_1\eta^{-2}C_4-\frac{1}{2}\varepsilon|\eta|^{-\frac{1}{2}}g(x_1^2\eta^{-2}-1),$$

$$H_2=|\eta|^{-\frac{3}{2}}(C_1+\frac{1}{4}F)-x_1\eta^{-2}C_3-\frac{1}{2}\varepsilon|\eta|^{-\frac{1}{2}}f(x_1^2\eta^{-2}+1),$$

$$E_3=\eta^{-1}C_2+x_1|\eta|^{-\frac{3}{2}}f,\quad H_3=\eta^{-1}C_4+x_1|\eta|^{-\frac{3}{2}}g$$

Here $f=f(\omega),\ g=g(\omega),\ F=F(\omega),\ G=G(\omega)$ are arbitrary smooth functions, $\dfrac{dF}{d\omega}=f,\ \dfrac{dG}{d\omega}=g,\ \omega=\xi-x_1^2\eta^{-1},$

$\xi=x_0+x_3,\quad \eta=x_0-x_3,$

$$\varepsilon=\begin{cases}1, \text{ if } x_0-x_3>0,\\ -1, \text{ if } x_0-x_3<0.\end{cases}$$

2) $0<|\alpha|\le 1$

$$E_1=x_3|\sigma|^{-\frac{3}{2}}C_1+C_2\eta^{\frac{2\alpha-1}{1-\alpha}},\ E_2=x_0|\sigma|^{-\frac{3}{2}}C_3+C_4\eta^{\frac{2\alpha-1}{1-\alpha}},$$

$$E_3=-x_1|\sigma|^{-\frac{3}{2}}C_1,$$

$$H_1=-x_3|\sigma|^{-\frac{3}{2}}C_3-C_4\eta^{\frac{2\alpha-1}{1-\alpha}},\ H_2=x_0|\sigma|^{-\frac{3}{2}}C_1+C_2\eta^{\frac{2\alpha-1}{1-\alpha}},$$

$$H_3=x_1|\sigma|^{-\frac{3}{2}}C_3.$$

If $\alpha=1$, then $C_2=C_4=0$.

Here $\sigma=x_0^2-x_1^2-x_3^2,\ \eta=x_0-x_3$.

M_8 :

$$E_1=-x_1\eta^{-2}C_1+\frac{1}{4}|\eta|^{-\frac{3}{2}}C_2(\xi+2\eta-3x_1^2\eta^{-1}+\ln|\eta|)+|\eta|^{-\frac{3}{2}}C_3,$$

$$E_2=x_1\eta^{-2}C_4-\frac{1}{4}|\eta|^{-\frac{3}{2}}C_5(\xi-2\eta-3x_1^2\eta^{-1}+\ln|\eta|)+|\eta|^{-\frac{3}{2}}C_6,$$

$$H_1=-x_1\eta^{-2}C_4+\frac{1}{4}|\eta|^{-\frac{3}{2}}C_5(\xi+2\eta-3x_1^2\eta^{-1}+\ln|\eta|)-|\eta|^{-\frac{3}{2}}C_6,$$

$$H_2=-x_1\eta^{-2}C_1+\frac{1}{4}|\eta|^{-\frac{3}{2}}C_2(\xi-2\eta-3x_1^2\eta^{-1}+\ln|\eta|)+|\eta|^{-\frac{3}{2}}C_3,$$

$E_3=\eta^{-1}C_1+x_1|\eta|^{-\frac{3}{2}}C_2,\ H_3=\eta^{-1}C_4+x_1|\eta|^{-\frac{3}{2}}C_5,$
where $\xi=x_0+x_3,\ \eta=x_0-x_3.$

M_9 : 1) $E_1=\varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2-4)-12\eta\omega)-\eta B_{12}],$
$E_2=\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2+4)-12\eta\omega)-\eta B_{34}],$
$E_3=\varphi^{-2}[4A_{12}(\eta\varphi^{-\frac{1}{2}}+6\omega)+2B_{12}],$
$H_1=-\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2-4)-12\eta\omega)-\eta B_{34}],$
$H_2=\varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2+4)-12\eta\omega)-\eta B_{12}],$
$H_3=-\varphi^{-2}[4A_{34}(\eta\varphi^{-\frac{1}{2}}+6\omega)+2B_{34}],$
where $A_{ij}=(1+36\omega^2)^{-\frac{3}{2}}[C_i\sigma^{\frac{1}{3}}(4\sqrt{1+36\omega^2}-72\omega)$
$+C_j\sigma^{-\frac{1}{3}}(4\sqrt{1+36\omega^2}+72\omega)],$
$B_{ij}=16(1+36\omega^2)^{-\frac{1}{2}}(C_i\sigma^{\frac{1}{3}}-C_j\sigma^{-\frac{1}{3}}),$
$\sigma=6\omega+\sqrt{36\omega^2+1},\ \omega=(\xi-x_1\eta+\frac{1}{6}\eta^3)\varphi^{-\frac{3}{2}},$
$\varphi=4x_1-\eta^2>0,$
$\xi=x_0+x_3,\ \eta=x_0-x_3;$
2) $E_1=\varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2+4)+42\eta\omega)-\eta B_{12}],$
$E_2=\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2-4)-42\eta\omega)-\eta B_{34}],$
$E_3=-\varphi^{-2}[4A_{12}(\eta\varphi^{-\frac{1}{2}}+21\omega)-2B_{12}],$
$H_1-\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2+4)+42\eta\omega)-\eta B_{34}],$
$H_2=\varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2-4)+42\eta\omega)-\eta B_{12}],$
$H_3=\varphi^{-2}[A_{34}(\eta\varphi^{-\frac{1}{2}}+21\omega)-2B_{34}],$
where $A_{ij}=(1-36\omega^2)^{-\frac{3}{2}}\{\cos\sigma[72\omega C_j-4C_i\sqrt{1-36\omega^2}]$
$-\sin\sigma[72\omega C_i+4C_j\sqrt{1-36\omega^2}]\},$
$B_{ij}=16(1-36\omega^2)^{-\frac{1}{2}}[C_i\sin\sigma-C_j\cos\sigma],\ \sigma=\frac{1}{3}\arcsin 6\omega,$
$|6\omega|<1,\ \varphi=\eta^2-4x_1>0,\ \omega=(\xi-x_1\eta+\frac{1}{6}\eta^3)\varphi^{-\frac{3}{2}},$
$\xi=x_0+x_3,\ \eta=x_0-x_3;$
3) $E_1=\varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2+4)-12\eta\omega)-\eta B_{12}],$
$E_2=\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}}-\varphi^{-\frac{1}{2}}(\eta^2-4)-12\eta\omega)-\eta B_{34}],$
$E_3=\varphi^{-2}[-4A_{12}(\eta\varphi^{-\frac{1}{2}}-6\omega)+2B_{12}],$

$$H_1 = -\varphi^{-2}[A_{34}(\varphi^{\frac{1}{2}} - \varphi^{-\frac{1}{2}}(\eta^2+4) - 12\eta\omega) - \eta B_{34}],$$
$$H_2 = \varphi^{-2}[A_{12}(\varphi^{\frac{1}{2}} - \varphi^{-\frac{1}{2}}(\eta^2-4) - 12\eta\omega) - \eta B_{12}],$$
$$H_3 = \varphi^{-2}[4A_{34}(\eta\varphi^{-\frac{1}{2}} - 6\omega) - 2B_{34}],$$
where $A_{ij} = (36\omega^2-1)^{-\frac{3}{2}}[C_i\sigma^{\frac{1}{3}}(4\sqrt{36\omega^2-1} - 72\omega)$
$+C_j\sigma^{-\frac{1}{3}}(4\sqrt{36\omega^2-1} + 72\omega)],$
$$B_{ij} = 16(36\omega^2-1)^{-\frac{3}{2}}[C_i\sigma^{\frac{1}{3}} - C_j\sigma^{-\frac{1}{3}}],$$
$$\sigma = 6\omega + \sqrt{36\omega^2-1},\quad |6\omega| > 1,\ \varphi = 4x_1 - \eta^2 > 0,$$
$$\omega = (\xi - x_1\eta + \frac{1}{6}\eta^3)\varphi^{-\frac{3}{2}},\ \xi = x_0 + x_3,\ \eta = x_0 - x_3.$$

M_{10} :
$$E_1 = \sigma^{-1}(1+\xi^2)^{-1}\{x_1C_5 - x_2C_6 - (1+\omega^2)^{-1}[\xi x_1(C_1\omega + C_2) + \xi x_2(C_3\omega + C_4) - \frac{1}{2}(1-\xi^2)(x_1(C_1 - \omega C_2) + x_2(C_3 - \omega C_4))]\} + \frac{1}{2}\sigma^{-2}(1+\xi^2)(1+\omega^2)^{-1}[x_1(C_1 - \omega C_2) - x_2(C_3 - \omega C_4)],$$
$$E_2 = \sigma^{-1}(1+\xi^2)^{-1}\{x_1C_6 + x_2C_5 + (1+\omega^2)^{-1}[\xi x_1(C_3\omega + C_4) - \xi x_2(C_1\omega + C_2) + \frac{1}{2}(1-\xi^2)(x_2(C_1 - \omega C_2) - x_1(C_3 - \omega C_4))]\} + \frac{1}{2}\sigma^{-2}(1+\xi^2)(1+\omega^2)^{-1}[x_1(C_3 - \omega C_4) + x_2(C_1 - \omega C_2)],$$
$$E_3 = \sigma^{-1}(1+\omega^2)^{-1}[C_1(\omega+\xi) + C_2(1-\xi\omega)],$$
$$H_1 = -\sigma^{-1}(1+\xi^2)^{-1}\{x_1C_6 + x_2C_5 + (1+\omega^2)^{-1}[\xi x_1(C_3\omega + C_4) - \xi x_2(C_1\omega + C_2) + \frac{1}{2}(1-\xi^2)(x_2(C_1 - \omega C_2) - x_1(C_3 - \omega C_4))]\} + \frac{1}{2}\sigma^{-2}(1+\xi^2)(1+\omega^2)^{-1}[x_1(C_3 - \omega C_4) + x_2(C_1 - \omega C_2)],$$
$$H_2 = \sigma^{-1}(1+\xi^2)^{-1}\{x_1C_5 - x_2C_6 - (1+\omega^2)^{-1}[\xi x_1(C_1\omega + C_2) + \xi x_2(C_3\omega + C_4) - \frac{1}{2}(1-\xi^2)(x_1(C_1 - \omega C_2) + x_2(C_3 - \omega C_4))]\} - \frac{1}{2}\sigma^{-2}(1+\xi^2)(1+\omega^2)^{-1}[x_1(C_1 - \omega C_2) - x_2(C_3 - \omega C_4)],$$
$$H_3 = \sigma^{-1}(1+\omega^2)^{-1}[C_3(\omega+\xi) + C_4(1-\xi\omega)],$$
where $\sigma = x_1^2 + x_2^2,\ \omega = \eta(1+\xi^2)\sigma^{-1} - \xi,\ \eta = x_0 + x_3,$
$\xi = x_0 - x_3.$

In the above formulae C_j, $(j = 1, 2, \ldots, 6)$ are arbitrary real constants.

Note that the constructed Maxwell fields are, generally speaking, non-orthogonal. However, provided some additional restrictions on the parameters $C_1, \ldots, C_6$ are imposed, they become orthogonal. Consider, as an example, the last solution from the above list. Imposing the orthogonality condition $\mathbf{E} \cdot \mathbf{H} = 0$ yields the following restrictions on the choice of $C_1, \ldots, C_6$:

$$C_2C_6 = C_4C_5, \quad C_1C_6 = C_1C_3 + C_2C_4 + C_3C_5.$$

Next, for the solution invariant under the subalgebra M_1 the orthogonality condition leads to the following set of algebraic equations to be satisfied by the parameters $C_1, \ldots, C_6$

$$C_2C_3 = C_1C_4, \quad C_1C_3 = 0.$$

6 Concluding remarks

The range of applications of the Lie group methods for solving systems of linear and nonlinear partial differential equations is so wide that it is simply impossible to give a detailed account of all the available techniques, even, if we restrict our considerations to some fixed group, like the conformal group $C(1, 3)$. However, the basic ideas and methods exposed in the present review paper are easily adapted to the cases of other groups of importance for the modern physics. In particular, it is straightforward to modify the general reduction method suggested here in order to make it applicable for solving equations of non-relativistic physics, where the central role is played by the Galilei and Schrödinger groups.

Furthermore, the general method exposed in the paper applies directly to solving the full Maxwell equations with currents. It can be used also to construct exact classical solutions of the Yang-Mills equations with Higgs fields and of their generalizations. Generically, the method developed in the paper can be efficiently applied to any conformally-invariant wave equation, on the solution set of which a covariant representation of the conformal algebra (2.11) is realized.

We do not consider here the solution techniques based on the symmetry reduction of different versions of the self-dual Yang-Mills equations to integrable models (we refer the interested reader to the papers [13]–[15], [22]–[24], [65] for a detailed exposition of the results in this field available by now).

The results on exact solution of nonlinear generalizations of the Maxwell equations are also beyond the scope of the present review. The survey of these results, as well as, the extensive list of references can be found in [21].

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